CMSC 250: Number Bases

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1 Introduction and Base 10

What does it mean to use base 10? It means that when we write a number, such as 639, the 9 is in the 1s place, the 3 is in the 10s place, and the 6 is in the 100s place.

If we think of $1 = 10^0$, $10 = 10^1$, and $100 = 10^2$ then we are saying:

$$639 = 6 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0$$

In general a number in base 10 has the expression:

$$n \text{ is written as } ...d_3d_2d_1d_0$$

Where $0 \leq d_i \leq 9$ and:

$$n = ... + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10^1 + d_0 \cdot 10^0$$

When we want to be completely clear that $...d_3d_2d_1d_0$ is in base 10 we'll subscript the number with 10, for example:

$$639_{10}$$

2 Base 8 (Octal)

In general a number in base 8 has the expression:

$$n \text{ is written as } ...d_3d_2d_1d_0$$

Where $0 \leq d_i \leq 7$ and:

$$n = ... + d_3 \cdot 8^3 + d_2 \cdot 8^2 + d_1 \cdot 8^1 + d_0 \cdot 8^0$$

Example 2.1. For example the expression 462 in base 8 represents the number:

$$462 = 4 \cdot 8^2 + 6 \cdot 8^1 + 2 \cdot 8^0 = 306_{10}$$

□

When we want to be completely clear that $...d_3d_2d_1d_0$ is in base 8 we'll subscript the number with 8, for example the above would be:

$$462_8 = 306_{10}$$

3 Base b

In general a number in base b has the expression:

$$n \text{ is written as } ...d_3 d_2 d_1 d_0$$

Where $0 \leq d_i \leq b-1$ and:

$$n = ... + d_3 \cdot b^3 + d_2 \cdot b^2 + d_1 \cdot b^1 + d_0 \cdot b^0$$

Example 3.1. For example the expression 235 in base 6 represents the number:

$$235_6 = 2 \cdot 6^2 + 3 \cdot 6^1 + 5 \cdot 6^0 = 85_{10}$$

□

Observe that when $b > 10$ the "digits" would need to be 10 and larger. Typically what is done is that after 9 we continue with A, B, C, etc.

4 Base 16 (Hexadecimal)

In general a number in base 16 has the expression:

$$n \text{ is written as } ...d_3 d_2 d_1 d_0$$

Where $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$ and we have the following, where A is interpreted as 10, B as 11, and so on:

$$n = ... + d_3 \cdot 16^3 + d_2 \cdot 16^2 + d_1 \cdot 16^1 + d_0 \cdot 16^0$$

Example 4.1. For example the expression $4EA$ in base 16 represents the number:

$$4EA_{16} = 4 \cdot 16^2 + 14 \cdot 16^1 + 10 \cdot 16^0 = 1258_{10}$$

□

5 Base 2 (Binary)

In general a number in base 2 has the expression:

$$n \text{ is written as } ...d_3 d_2 d_1 d_0$$

Where $0 \leq d_i \leq 1$ and:

$$n = ... + d_3 \cdot 2^3 + d_2 \cdot 2^2 + d_1 \cdot 2^1 + d_0 \cdot 2^0$$

Example 5.1. For example the expression 110101 in base 2 represents the number:

$$110101_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 53_{10}$$

□

6 Base 10 to Base b the Slow Way

We've seen how to convert from base b to decimal. To convert from decimal to base b. Given a decimal number n, the most obvious way is to start by finding the largest power of b, say b^i, which is less than n, then count how many of those you can subtract from n still keeping a nonnegative result. That value is your leftmost digit.

Next, repeat the process with b^{i-1}, b^{i-2}, and so on, adding the resulting digits to the right until a result of 0 is obtained.

Example 6.1. Let's convert 9789_{10} to base 8. First we observe that $8^4 = 4096$ and $8^5 = 32768$ so $8^4 = 4096$ is the largest. We can subtract two of these:

$$9789 - 2(4096) = 1597$$

Thus our leftmost digit is 2.

Then we have 1597 left. From this we can subtract three $8^3 = 512$s:

$$1597 - 3(512) = 61$$

Thus our next digit is 3.

Then we have 61 left. From this we can subtract zero $8^2 = 64$s. Thus our next digit is 0.

Then we still have 61 left. From this we can subtract seven $8^1 = 8$s:

$$61 - 7(8) = 5$$

Thus our next digit is 7.

Now we have 5 left so the final digit is 5.

Thus:

$$9789_{10} = 23075_8$$

□

7 Base 10 to Base b the Fast Way

There's a slicker, more organized way to go about this. Let's to back to our example of 9789 which we'd like in base 8.

What we do is start with 9789 and repeatedly divide by 8, each successive time using the previous quotient as the dividend and continuing until the quotient is 0:

$$\begin{aligned}
9789 \div 8 &= 1223\,R\,5 \\
1223 \div 8 &= 152\,R\,7 \\
152 \div 8 &= 19\,R\,0 \\
19 \div 8 &= 2\,R\,3 \\
2 \div 8 &= 0\,R\,2
\end{aligned}$$

We then put down the remainders in reverse order:

$$9789_{10} = 23075_8$$

This looks a bit like magic so it's worth taking a second to see what this calculation has actually told us. If we rewrite the successive divisions above:

$$\begin{aligned}
9789 &= 1223 \cdot 8 + 5 \\
&= (152 \cdot 8 + 7) \cdot 8 + 5 \\
&= 152 \cdot 8^2 + 7 \cdot 8 + 5 \\
&= (19 \cdot 8 + 0) \cdot 8^2 + 7 \cdot 8 + 5 \\
&= 19 \cdot 8^3 + 0 \cdot 8^2 + 7 \cdot 8 + 5 \\
&= (2 \cdot 8 + 3) \cdot 8^3 + 0 \cdot 8^2 + 7 \cdot 8 + 5 \\
&= 2 \cdot 8^4 + 3 \cdot 8^3 + 0 \cdot 8^2 + 7 \cdot 8 + 5
\end{aligned}$$

This is exactly the expansion we wanted!

Note that when we convert to hexadecimal we have to keep an eye on our digits A through F.

> **Example 7.1.** Let's convert 20075 to hexadecimal.

$$
\begin{aligned}
20075 \div 16 &= 1254\,R\,11 \\
1254 \div 16 &= 78\,R\,6 \\
78 \div 16 &= 4\,R\,14 \\
4 \div 16 &= 0\,R\,4
\end{aligned}
$$

The "digits" 4, 14, 6, and 11 become $4E6B_{16}$.

□

8 Binary to Octal and Hexadecimal by Grouping

In the special case when we wish to convert between binary and either octal or hexadecimal we can do so quickly by grouping. This works specifically because 8 and 16 are powers of 2.

To convert to octal simply group the bits into groups of three and convert each group directly to octal. Make sure to group from right to left and prepend 0s if necessary.

Example 8.1. Let's convert 11010100001110_2 to octal. We group them and prepend a single 0 and convert:

Binary	011	010	100	001	110
Octal	3	2	4	1	6

Thus the answer is 32416_8.

□

To convert to hexadecimal simply group the bits into groups of four and convert each group directly to octal. Make sure to group from right to left and prepend 0s if necessary. We may have to think of an intermediate decimal value for each group but the values are small.

Example 8.2. Let's convert 11110101101110_2 to hexadecimal. We group them and prepend two 0s and convert:

Binary	0011	1101	0110	1110
Decimal	3	13	6	14
Hexadecimal	3	D	6	E

Thus the answer is $3D6E_{16}$.

□

9 Octal and Hexadecimal to Binary by Grouping

In these cases we can simply reverse the above procedure.

Example 9.1. Let's convert 5610_8 to binary. We simply convert 5, 6, 1, and 0 independently and string them together, yielding 101, 110, 001, and 000 so then 101110001000_2.

□

Example 9.2. Let's convert $A5D9_{16}$ to binary. We simply convert A, 5, D, and 9 independently and string them together, yielding 1010, 0101, 1110, and 1001 so then 1010010111101001_2.

□

10 Base n Addition

We add in base n just as we do in decimal. When we add digits we have to make sure that the resulting number is in the correct base and we must carry when appropriate. This requires that we can convert relatively small numbers into the necessary base but this is usually pretty easy to do.

Example 10.1. Let's do $234_5 + 403_5$. We line them up:

$$\begin{array}{ccc} 2 & 3 & 4 \\ 4 & 0 & 3 \end{array}$$

We add $4 + 3 = 7$ and in base 5 this is 12 so we put down the 2 and carry the 1:

$$\begin{array}{ccc} & 1 & \\ 2 & 3 & 4 \\ 4 & 0 & 3 \\ \hline & & 2 \end{array}$$

Then we add $1 + 3 + 0 = 4$ and in base 5 this is still 4 so we put down the 4:

$$\begin{array}{ccc} & 1 & \\ 2 & 3 & 4 \\ 4 & 0 & 3 \\ \hline & 4 & 2 \end{array}$$

Then we add $2 + 4 = 6$ and in base 5 this is 11 so we put that down:

$$\begin{array}{cccc} & & 1 & \\ & 2 & 3 & 4 \\ & 4 & 0 & 3 \\ \hline 1 & 1 & 4 & 2 \end{array}$$

Thus:

$$234_5 + 403_5 = 1142_5$$

$\square$

When we do hexadecimal we have to be careful with our sums because of A through F.

Example 10.2. Consider $7F2E_{16} + BD5C_{16}$. We line them up:

$$\begin{array}{cccc} 7 & F & 2 & E \\ B & D & 5 & C \end{array}$$

We add $E + C = 1A$ in hexadecimal so we put down the A and carry the 1:

$$\begin{array}{cccc} & & 1 & \\ 7 & F & 2 & E \\ B & D & 5 & C \\ \hline & & & A \end{array}$$

Then we add $1 + 2 + 5 = 8$ in hexadecimal so we put down the 8:

$$\begin{array}{cccc} & & 1 & \\ 7 & F & 2 & E \\ B & D & 5 & C \\ \hline & & 8 & A \end{array}$$

Then we add $F + D = 1C$ in hexadecimal so we put down the C and carry the 1:

$$\begin{array}{cccc} 1 & & 1 & \\ 7 & F & 2 & E \\ B & D & 5 & C \\ \hline & C & 8 & A \end{array}$$

Then we add $1 + 7 + B = 13$ in hexadecimal so we put down both:

$$\begin{array}{ccccc} & 1 & & 1 & \\ & 7 & F & 2 & E \\ & B & D & 5 & C \\ \hline 1 & 3 & C & 8 & A \end{array}$$

The result is then $13C8A_{16}$.

$\square$

Binary is a little easier since there are only a few possible sums. Here's an example without all the details:

Example 10.3. Here is $1100101011_2 + 0110110011_2$.

$$
\begin{array}{ccccccccccc}
 & 1 & & & 1 & & & & 1 & 1 & \\
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

$\square$