CMSC 250: Events and Basic Counting

Justin Wyss-Gallifent

April 25, 2023

1	Introduction		2
	1.1	Events	2
	1.2	Counting Outcomes	2
2	The Addition Rule for Disjoint Events		3
	2.1	Definition	3
	2.2	Rule	3
	2.3	Examples	3
3	The Multiplication Rule for Independent Events		4
	3.1	Definition	4
	3.2	Rule	4
	3.3	Examples	4
4	The Subtraction Rule for Complements of Events		5
	4.1	Introduction	5
	4.2	Rule	5

1 Introduction

The goal of this section is to discuss a number of basic combinatorical rules. These can be thought of essentially as rules of counting.

1.1 Events

Definition 1.1.1. The term *event* is loosely used to describe something that occurs for which there is some sort of *outcome*. The outcome could be numerical, or something else. More rigorously an event can be thought of as a choice of outcomes from a set.

> **Example 1.1.** Rolling a die is an event which yields an outcome in the set $\{1, 2, 3, 4, 5, 6\}$.

> **Example 1.2.** Rolling two distinct dice is an event which yields an outcome in the set:
> $$\{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x \leq 6, 1 \leq y \leq 6\}$$

> **Example 1.3.** Flipping a coin is an event which yields an outcome in the set $\{H, T\}$.

> **Example 1.4.** Earning a grade in this class is an event which yields an outcome in the set $\{A, B, C, D, F\}$.

> **Example 1.5.** A brewery has light beer or dark beer. You order two beers. This choice is an event which yields an outcome in the set $\{LL, DD, LD\}$.

Note 1.1.1. There do not need to be a finite number of outcomes in an event but for all our counting problems there will be.

Note 1.1.2. An event can be made up of smaller events.

> **Example 1.6.** Suppose you must choose between flipping a coin and rolling a die, Each of these can be thought of as its own event with outcomes $\{H, T\}$ and $\{1, 2, 3, 4, 5, 6\}$ respectively, or we can think of your choice as one event which yields an outcome in the set $\{H, T, 1, 2, 3, 4, 5, 6\}$.

1.2 Counting Outcomes

In this section we are interested in counting the number of outcomes in an event. We may be interested in counting all of the outcomes or just some of them.

Example 1.7. If we roll a die there are 6 outcomes: $\{1, 2, 3, 4, 5, 6\}$

Example 1.8. If we roll a die but are only interested in values greater than 2 then there are 4 outcomes: $\{3, 4, 5, 6\}$

Example 1.9. If we roll two distinct dice and are only interested in the outcomes which add to 10 then there are 3 outcomes: $\{(4, 6), (5, 5), (6, 4)\}$

2 The Addition Rule for Disjoint Events

2.1 Definition

Definition 2.1.1. Two or more events are *disjoint* if only one of them can occur.

Example 2.1. Suppose you must choose between flipping a coin and rolling a die, as in the the previous example. These two events are disjoint.

2.2 Rule

Theorem 2.2.1. Suppose there are k disjoint events and event i has n_i outcomes. If exactly one of the k events occurs then the total number of outcomes is:

$$n_1 + n_2 + ... + n_k$$

2.3 Examples

Example 2.2. If we have a red die and a green die but we only get to roll one of them then there are $6 + 6 = 12$ outcomes.

Example 2.3. Suppose there are n buckets labeled 1, 2, ..., n. In bucket i there are i items. We get to select one item from one bucket. How many ways can we do this? Well, in total the number of outcomes will be:

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

Example 2.4. If we pick a letter or a decimal digit (but not both), and the letter is not A, then there are $25 + 10 = 35$ outcomes.

3 The Multiplication Rule for Independent Events

3.1 Definition

Definition 3.1.1. Two or more events are *independent* if the outcome of one of them does not affect the outcome of the other(s).

Example 3.1. Suppose you flip a coin and then roll a die. These two events are independent since the outcomes from the coin flip and the outcome from the die roll do not affect one another.

3.2 Rule

Theorem 3.2.1. Suppose there are k independent events and event i has n_i outcomes. If all of the k events occurs then the total number of outcomes is:

$$n_1 n_2 ... n_k$$

3.3 Examples

Example 3.2. If we roll two different dice then there are $(6)(6) = 36$ outcomes.

Example 3.3. If we flip two different coins and roll one die then there are $(2)(2)(6) = 24$ outcomes.

Example 3.4. A store sells three different iPhones and four different Pixels. If we buy an iPhone and a Pixel then there are $(3)(4) = 12$ outcomes.

Note 3.3.1. At this point it's worth noting why the word "different" (or a similar word) is used a lot. When we are dealing with two (or more) of the same type of thing (two dice, two pizzas, two coins, etc.) it is important to fully understand what is being counted.

Other words and phrases which are often used are things like "order matters" and "distinct". Or it can be made clear that the two (or more) things are for two or more people (a pizza for Jack and a pizza for Jill) and so on.

Example 3.5. If we roll two distinct dice then there are 36 outcomes. The word "distinct" is relevant because we consider $(1, 6)$ and $(6, 1)$ to be different, because the dice are distinct.

However if we simply roll two dice the arguably $(1, 6)$ and $(6, 1)$ might be considered the same. In such a case the set of outcomes are:

$$(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$$

$$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)$$

This is only 21 outcomes.

4 The Subtraction Rule for Complements of Events

4.1 Introduction

For certain events, sometimes it's easier to count the outcomes which we're not interested in and subtract that from the total number of outcomes,

Example 4.1. Suppose we roll two distinct dice. How many outcomes will not yield a sum of 10?

It's easier to count the number of outcomes which do yield a sum of 10.

4.2 Rule

Theorem 4.2.1. Suppose there are n outcomes and we are interested in those which fulfill some criteria. If k of them do not fulfill that criteria then $n-k$ of them do.

Example 4.2. Suppose we roll two distinct dice. How many outcomes will not yield a sum of 10?

There are $(6)(6) = 36$ total outcomes from rolling two distinct dice. The outcomes which do add to 10 are $\{(4,6),(5,5),(6,4)\}$. Since there are 3 of these which do we know $36-3$ do not.

Example 4.3. Suppose we flip five coins in a row. How many of them don't start with HH?

Well there are $(2)(2)(2)(2)(2) = 32$ outcomes all together. Of these, the ones that start with HH look like HH??? and there are $(2)(2)(2) = 8$ of these. Therefore there are $32-8 = 24$ ones which don't start with HH.