CMSC 250: First Order Logic - Quantification

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1 A Tiny Bit of Set Notation

Even though we'll discuss set notation at length later there is a small collection of symbols which will make our life easier right now.

Definition 1.0.1. The symbol $\in$ means "is an element in the set of (whatever follows)"

Here are some examples:

Example 1.1. We would write:

(a) $x \in \{1, 2, 3\}$ to say that x is one of the three numbers 1,2, or 3.

(b) $x \in \{\text{primes}\}$ to say that x is a prime number.

Definition 1.0.2. The symbol $\mathbb{Z}$ means the set of all integers and the symbol $\mathbb{R}$ means the set of all real numbers.

Thus:

Example 1.2. We would write:

(a) $x \in \mathbb{Z}$ to say that x is an integer.

(b) $x \in \mathbb{R}$ to say that x is a real number.

Definition 1.0.3. The list goes on; We also have $\mathbb{Q}$ for rational numbers, $\mathbb{C}$ for complex numbers, $2\mathbb{Z}$ for even integers, and lots more.

2 Quantifying Variables

2.1 Introduction

We saw in propositional logic that we cannot have variables in our statements unless we clarify what they are, otherwise it's impossible to assign a truth value.

Example 2.1. The sentence "p is an odd prime number" is not a statement because don't know the value of p. It might be $p = 2$, in which case the statement is false, otherwise the statement is true.

There are two classic ways to take a sentence which includes one or more variables and turn it into a statement.

2.2 Universal; For All

Suppose $P(x)$ is some sentence which involves a variable x such that when a particular x is plugged in, $P(x)$ becomes a statement. That x could be a number or even a more abstract thing.

For example, suppose $P(x)$ is the sentence "x weighs less than one ton". Then for example, if x is "Justin" then the statement is true, whereas if x is "A Boeing 737" then the statement is false.

Consider now the sentence:

For all humans x, x weighs less than one ton.

We can see that this sentence is actually a statement and it is true. On the other hand consider the sentence:

For all airplanes x, x weighs less than one ton.

This sentence is also a statement and it is false, since some airplanes weigh a ton or more.

What we'll do now is introduce some notation.

Definition 2.2.1. The quantifier $\forall$ is read "for all".

Using this notation, then our previous examples become:

- The statement
$$\forall x \in \{\text{humans}\}, x \text{ weighs less than one ton}$$
is true.
- The statement
$$\forall x \in \{\text{airplanes}\}, x \text{ weighs less than one ton}$$
is false.

Here are some more:

Example 2.2. The statement:
$$\forall x \in \mathbb{R}, x^2 \geq 0$$
is true.

Example 2.3. The statement:
$$\forall x \in \mathbb{R}, x^2 > 0$$
is false, since $x = 0$ makes it false, so it's not true for all reals.

Example 2.4. If $D = \{2, 3, 5, 7\}$ then the statement:
$$\forall x \in D, x \text{ is prime}$$
is true.

2.3 Existential; There Exists

To balance this out we'll also have the notation:

Definition 2.3.1. The quantifier $\exists$ is read "there exists" and will make the statement true if there is at least one value of the variable which makes it true.

Note 2.3.1. The comma used is then often read as "such that". For example

Using this notation, here are examples similar to our previous.

- The statement

  $$\exists x \in \{\text{humans}\}, x \text{ weighs less than 100 pounds}$$

  is true. This is read as:

  "There is an x in the set of humans such that x weighs less than 100 pounds."
  or "There is a human who weighs less than 100 pounds."

- The statement

  $$\exists x \in \{\text{humans}\}, x \text{ weighs more than one ton}$$

  is false.

Here are some more:

Example 2.5. The statement:

$$\exists x \in \mathbb{Z}, x \text{ is prime}$$

is true.

Example 2.6. The statement:

$$\exists x \in \mathbb{Z}, 3x = 12$$

is true.

Example 2.7. The statement:

$$\exists x \in \mathbb{Z}, 3x = 7$$

is false.

However:

Example 2.8. The statement:

$$\exists x \in \mathbb{R}, 3x = 7$$

is true, as the integer $x = \frac{7}{3}$ works.

2.4 Multiple Quantifiers

It's certainly possible to have a sentence which involves more than one variable. In such a case we may introduce one quantifier for each variable.

When the quantifiers are the same the order does not matter and the statement is usually fairly easy to understand.

Example 2.9. Consider the statement:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 100$$

This statement is saying that when x and y are integers the sum is always 100. This statement is clearly false. Moreover we could have written $\forall y \in \mathbb{Z}, \forall x \in \mathbb{Z}$ instead with no difference.

Example 2.10. Consider the statement:

$$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y = x^2$$

This statement is saying that it's possible to find integers x and y which satisfy $y = x^2$. This statement is clearly true, for example $x = 5$ and $y = 25$. Moreover we could have written $\exists y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ instead with no difference.

Example 2.11. Consider the statement:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x^2 + y^2 \geq 0$$

This statement is saying that when x and y are integers the sum of the squares is always greater than or equal to 0. This statement is clearly true. Moreover we could have written $\forall y \in \mathbb{Z}, \forall x \in \mathbb{Z}$ instead with no difference.

Example 2.12. Consider the statement:

$$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = \frac{1}{2}$$

This statement is saying that it's possible to find integers x and y whose sum is $\frac{1}{2}$. This statement is clearly false. Moreover we could have written $\exists y \in \mathbb{Z}, \exists x \in \mathbb{Z}$ instead with no difference.

Note 2.4.1. When the quantifiers and the sets are the same we can shorthand the notation. For example instead of writing $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}$ we can simply write $\forall x, y \in \mathbb{Z}$.

However when the quantifers are different then the sentences will generally mean entirely different things.

Consider the following real-world examples. For the sake of argument we're only considering regular currencies here, not things like Bitcoin.

> **Example 2.13.** Consider the statement:
>
> For every country, there is a currency such that the currency works in the country.
>
> More formally we might write:
>
> $$\forall x \in \{\text{countries}\}, \exists y \in \{\text{currencies}\}, y \text{ works in } x$$

> **Example 2.14.** Consider the statement:
>
> There is a currency, for every country such that the currency works in the country.
>
> More formally we might write:
>
> $$\exists y \in \{\text{currencies}\}, \forall x \in \{\text{countries}\}, y \text{ works in } x$$

Stop and really think about the difference between these. The first one is saying that if you choose a country, any country, there is a currency which works in in that country. This is clearly true. On the other hand the second one is saying that there is a currency that works in every country. This is clearly false.

Consider now a mathematical pair:

> **Example 2.15.** The statement:
>
> $$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 2$$
>
> This is clearly true, since $y = 2 - x$ will work for any x that's given.

> **Example 2.16.** The statement:
>
> $$\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 2$$
>
> This is clearly false. The claim here that there is one single $y \in \mathbb{Z}$ which will work for every $x \in \mathbb{Z}$.

Lastly observe that the quantifiers don't need to be adjacent.

Example 2.17. Consider the statement:

$$\forall x \in \mathbb{R} \left[x \geq 0 \Longrightarrow \exists y \in \mathbb{R}, y^2 = x \right]$$

This is clearly true.

2.5 Informality and Implicit Quantification

It's not at all uncommon for us to be informal and/or implicit with our quantification.

An example of informality would be:

Example 2.18. Consider the statement:

Squaring a real number yields a result which is greater than or equal to zero.

This would be more formally put as:

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

Likewise when the context is clear we assume that it's understood.

Example 2.19. Suppose you're a number theorist communicating with another number theorist and you know for a fact that you only ever discuss integers. You might write something like:

$$\forall n, 2n \text{ is even}$$

It's understood that $n \in \mathbb{Z}$ and the statement is true. You would never write this to some general mathematician who would answer "No, it's false, for example $n = \frac{1}{2}$.

We strongly recommend making it clear!

2.6 Revisiting Parity Etc.

Just to close this section let's revisit our definitions related to parity, primes, divisibility, and modular arithmetic, using these quantifiers.

Definition 2.6.1. We say that an integer x is even iff $\exists k \in \mathbb{Z}, x = 2k$.

Definition 2.6.2. We say that an integer x is odd iff $\exists k \in \mathbb{Z}, x = 2k + 1$.

Definition 2.6.3. Given integers a, b with $a \neq 0$ we say that $a \mid b$ iff $\exists c \in \mathbb{Z}, ac = b$.

In the next definition, $\mathbb{Z}^+$ denotes the set of positive integers.

Definition 2.6.4. Given an integer $n \geq 2$ we say that n is prime iff $\forall d \in \mathbb{Z}^+, (d|n) \rightarrow ((d = 1) \vee (d = n))$.

Definition 2.6.5. Given an integer $n \geq 2$ we say that n is composite iff $\exists d \in \mathbb{Z}^+, (d|n) \wedge (1 < d < n)$.

Definition 2.6.6. Given integers a, b, m with $m \geq 2$ we say that $a \mod m = b$ iff $(0 \leq b < m) \wedge \exists q \in \mathbb{Z}, a = qm + b$.

Definition 2.6.7. Given integers a, b, m with $m \geq 2$ we say that $a \equiv b \mod m$ iff $m \mid (a - b)$ iff $\exists k \in \mathbb{Z}, km = a - b$

2.7 Unbound Variables

Definition 2.7.1. When we have a variable which is not bound, we say it is *unbound*.

Unbound variables are not really something we will work with, except when you forget to bind your variables and we say "Hey, this variable is unbound!"

3 Negating Quantifiers

Suppose I said to you:

Every human is under six feet tall!

You would probably tell me I'm false by arguing:

There is a human who is six feet tall or taller!

What happened here? You negated my statement, but let's look at a more formal version of this:

I said:

$$\forall x \in D, P(x)$$

You replied:

$$\exists x \in D, \sim P(x)$$

In essence we have:

$$\sim\big[\forall x \in D, P(x)\big] \equiv \exists x \in D, \sim P(x)$$

Notice what happened here. The general rule is as follows:

Rule: When negating a quantified statement, switch the quantifer from $\forall$ to $\exists$ or $\exists$ to $\forall$ and push the negation inside.

Note 3.0.1. Once the negation is inside it can be beneficial to do more tidying up inside.

Here are some abstract examples:

Example 3.1. Here is a negation involving a conjunction inside:

$$\begin{aligned}\sim\bigl[\exists x, P(x) \wedge Q(x)\bigr] &\equiv \forall x, \sim(P(x) \wedge Q(x)) \\ &\equiv \forall x, \sim P(x) \vee \sim Q(x)\end{aligned}$$

Example 3.2. Here is a negation involving an implication inside:

$$\begin{aligned}\sim\bigl[\forall x, P(x) \rightarrow Q(x)\bigr] &\equiv \exists x, \sim(P(x) \rightarrow Q(x)) \\ &\equiv \exists x, P(x) \wedge \sim Q(x)\end{aligned}$$

Example 3.3. Here is a negation involving two quantifiers and some stuff inside:

$$\begin{aligned}\sim\bigl[\exists x, \forall y, P(x) \vee (Q(x) \rightarrow \sim R(x))\bigr] &\equiv \forall x \exists y, \sim(P(x) \vee (Q(x) \rightarrow \sim R(x))) \\ &\equiv \forall x \exists y, \sim P(x) \wedge \sim(Q(x) \rightarrow \sim R(x)) \\ &\equiv \forall x \exists y, \sim P(x) \wedge (Q(x) \wedge R(x))\end{aligned}$$

Here are some more concrete ones. You can think about whether they're true or false but that's not really the issue here.

Example 3.4. We have:

$$\begin{aligned}\sim\bigl[\exists x \in \mathbb{Z}, x^2 = 20\bigr] &\equiv \forall x \in \mathbb{Z}, \sim(x^2 = 20) \\ &\equiv \forall x \in \mathbb{Z}, x^2 \neq 20\end{aligned}$$

Example 3.5. We have:

$$\begin{aligned}\sim\bigl[\forall n \in \mathbb{Z}, \text{if } n \text{ is odd then } n \text{ is prime}\bigr] &\equiv \exists n \in \mathbb{Z}, \sim(\text{if } n \text{ is odd then } n \text{ is prime}) \\ &\equiv \exists n \in \mathbb{Z}, n \text{ is odd and } n \text{ is not prime}\end{aligned}$$

4 Contrapositive, Converse and Inverse Revisited

Let's revisit the contrapositive, converse and inverse once more:

- The contrapositive of $P \rightarrow Q$ is $\sim Q \rightarrow \sim P$.
- The converse of $P \rightarrow Q$ is $Q \rightarrow P$.
- The inverse of $P \rightarrow Q$ is $\sim P \rightarrow \sim Q$

Since these are things which apply to implications, not to quantifiers per se, when we discuss these terms the quantifiers remain unchanged. For example:

- The contrapositive of $\forall x(P(x) \to Q(x))$ is $\forall x(\sim Q \to \sim P(x))$.
- The converse of $\exists x(P(x) \to Q(x))$ is $\exists x(Q(x) \to P(x))$.
- The inverse of $\forall x(P(x) \to Q(x))$ is $\forall x(\sim P(x) \to \sim Q(x))$

Here are some more specific examples:

Example 4.1. The contrapositive of:

$$\forall p \in \{\text{primes}\} \forall a \in \mathbb{Z}, \text{if } a \mid p \text{ then } (a = 1 \text{ or } a = p)$$

would be (note DeMorgan's Law):

$$\forall p \in \{\text{primes}\} \forall a \in \mathbb{Z}, \text{If } (a \neq 1 \text{ and } a \neq p) \text{ then } a \nmid p$$

Remember from earlier that the contrapositive of a statement is equivalent to the statement. In this case both of these are true.

Example 4.2. The converse of:

For all $a, b, m \in \mathbb{Z}$ with $m \geq 2$, if $a \equiv b \mod m$ then $a^2 \equiv b^2 \mod m$

would be:

For all $a, b, m \in \mathbb{Z}$ with $m \geq 2$, if $a^2 \equiv b^2 \mod m$ then $a \equiv b \mod m$

Remember from earlier that the converse of a statement is not equivalent to the statement. In this case the original statement is true but the converse is false.

Example 4.3. The inverse of:

$$\forall a \in \mathbb{Z}, \text{ if } a > 3 \text{ then } a^2 > 100$$

would be:

$$\forall a \in \mathbb{Z}, \text{ if } a \not> 3 \text{ then } a^2 \not> 100$$

or, better:

$$\forall a \in \mathbb{Z}, \text{ if } a \leq 3 \text{ then } a^2 \leq 100$$

Remember from earlier that the inverse of a statement is not equivalent to the statement. In this case the original statement is false but the converse is true.