CMSC 250: Permutations and Combinations

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April 27, 2023

1 Permutations of k Objects out of n Distinct

1.1 Introductory Example

Suppose we have five kittens and wish to select three of them and place them in order. When order matters this is called a *permutation*. In this case imagine three positions into which the kittens will go.

- Into the first position we have 5 kittens to choose from.
- Into the second position we have 4 kittens to choose from.
- Into the third position we have 3 kittens to choose from.

Since we are making all of these choices, the total number of possibilities is:

$$5 \cdot 4 \cdot 3 = 60$$

1.2 General Rule

Suppose we have n distinct items and wish to select k of them and place them in order. We say we are permuting k out of n items, or just "n permute k".

The total number of possibilities is:

$$P(n,k) = \frac{n!}{(n-k)!}$$

Note 1.2.1. This is sometimes denoted nPk.

Note 1.2.2. If we're permuting all of them then $k = n$ and this is just $n!$.

1.3 Basic Examples

Example 1.1. The number of ways to permute 5 out of 7 objects is:

$$P(7,5) = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520$$

Example 1.2. The number of ways to permute 5 objects is:

$$P(5,5) = 5! = 120$$

1.4 Sneaky Examples

Example 1.3. Suppose there are 7 people and a round table with 5 seats. We wish to seat 5 of the people. While the relative position of the people matter, the absolute position does not. How many ways can we do this?

The approach here is to first ignore the fact that the absolute position does not matter and just imagine we're positioning people at the table. There are $P(7,5) = 2520$ ways to do this. Now then, those 2520 can be grouped into groups of 5 where each group of 5 contains the same relative arrangement (the 5 rotations).

Therefore all together there are $2520/5 = 504$ possibilities.

Example 1.4. How many ways are there to permute the letters in PYTHON if the P and Y cannot be adjacent?

The approach here is to note that there are $P(6,6)$ ways to permute all of the letters and then count and subtract the total number of ways in which they are together.

- If they are adjacent in the order PY then there are five ways this could happen:

  PY????, ?PY???, ??PY??, ???PY?, and ????PY

  However for each of these there are four other letters to permute in the four remaining positions, so each has 4! possibilities. Thus there are 5(4!) possibilities which look like these.

- If they are adjacent in the order YP then there are five ways this could happen:

  YP????, ?YP???, ??YP??, ???YP?, and ????YP

  However again for each of these there are four other letters to permute in the four remaining positions, so each has 4! possibilities. Thus there are 5(4!) possibilities which look like these.

Thus all together the number of possibilities is:

$$P(6,6) - 5(4!) - 5(4!) = 720 - 120 - 120 = 480$$

2 Combinations of k Objects out of n Distinct

2.1 Introductory Example

Suppose we have five kittens and wish to select three of them and place them in a box. Don't worry, it's a large box, they have toys, and they're on their way to a good home.

When order does not matter this is called a *combination*. How many possible ways are there to do this?

The approach to finding a general rule is like the table example from the previous section. First let's consider if the order did matter. Then it's just $P(5,3)$.

However imagine now that certain permutations of kittens would be considered the same because we don't care about order. For example for kittens 1,2,5 we would have:

$$\{1,2,5\} = \{1,5,2\} = \{2,1,5\} = \{2,5,1\} = \{5,1,2\} = \{5,2,1\}$$

We see that all of the $P(5,3)$ permutations can be grouped into groups of 3! where each group of 3! contains the same collection.

Therefore all together the total number of possibilities is:

$$\frac{P(5,3)}{3!}$$

Notice that:

$$\frac{P(5,3)}{3!} = P(5,3) \cdot \frac{1}{3!} = \frac{5!}{(5-3)!} \cdot \frac{1}{3!} = \frac{5!}{(5-3)!3!}$$

2.2 General Rule

Suppose we have n distinct items and wish to select k of them and place them together where order does not matter. We say we are choosing k out of n items, or just "n choose k".

The total number of possibilities is:

$$C(n,k) = \frac{n!}{(n-k)!k!}$$

Note 2.2.1. This is sometimes denoted nCk or $\binom{n}{k}$.

2.3 Basic Examples

Example 2.1. The number of ways to choose 5 out of 7 objects is:

$$C(7,5) = \frac{7!}{(7-5)!5!} = \frac{7!}{2!5!} = 21$$

2.4 Sneaky Examples

Consider these more challenging examples.

Example 2.2. Suppose we have 12 people and wish to choose a team of 7. Order does not matter but there's an issue. Two of the people have stated that they must stick together, meaning they're either both on the team or both not. How many possibilities are there?

The approach here is to divide the possibilities into two disjoint sets, those with the two people and those without.

- If the two people are on the team then we only need to choose 5 more people out of the remaining 10 and there are $C(10,5)$ ways to do this.
- If the two people are not on the team then we need to choose 7 people out of the remaining 10 and there are $C(10,7)$ ways to do this.

Since we only get to do one of these we add.

The total number of possibilities is then:

$$C(10,5) + C(10,7) = \frac{10!}{(10-5)!5!} + \frac{10!}{(10-7)!7!} = ... = 372$$

3 Permutations of a Fixed Set with Repeated Elements

3.1 Introduction and Example

Suppose we wish to permute the letters in the word BABAR keeping in mind that the two As and the two Bs are indistinguishable. How many possibilities are there?

This is a permutation question (order matters) but we have repeated elements. The appoach here is to imagine that we have five positions which we wish to fill with the letters B, A, B, A, and R.

$$\_\_\_ \quad \_\_\_ \quad \_\_\_ \quad \_\_\_ \quad \_\_\_$$

First let's choose where to put the Bs. We need to choose two positions out of the five and there are $C(5,2)$ ways to do this. Notice that since we're choosing positions the order does not matter, since for example choosing positions 1,4 and putting a B in each is the same as choosing positions 4,1 and putting a B in each.

Second let's choose where to put the As. We need to choose two positions out of the remaining three and there are $C(3,2)$ ways to do this.

Third let's choose where to put the R. There's really no choice at all but we could say that we need to choose one position out of the remaining one and there are $C(1,1)$ ways to do this.

Since we do all of these, the total number of possibilities is then:

$$C(5,2)C(3,2)C(1,1)$$

Notice that as a calculation this simplifies:

$$C(5,2)C(3,2)C(1,1) = \frac{5!}{(5-2)!2!} \cdot \frac{3!}{(3-2)!2!} \cdot \frac{1!}{(1-1)!1!} = \frac{5!}{3!2!} \cdot \frac{3!}{1!2!} \cdot \frac{1!}{0!1!} = \frac{5!}{2!2!1!}$$

3.2 General Rule

Suppose we wish to permute n items which are of k types and:

- n_1 of one type
- n_2 of another type
- ...
- n_k of the last type

The number of distinguishable permutations is:

$$\frac{n!}{n_1!n_2!...n_k!}$$

3.3 Basic Examples

Example 3.1. How many distinct binary strings of length twelve containing eight 1s are there?

Here we are permuting $1,1,1,1,1,1,1,1,0,0,0,0$.

Thus $n = 12$, $n_1 = 8$ and $n_2 = 4$.

The total number of possibilities is then:

$$\frac{12!}{8!4!} = .. = 495$$

Example 3.2. Our local pizza shop has plain, mushroom, and bacon slices. We are hungry and want 8 slices of pizza, specifically we want 4 plain, 3 mushroom, and 1 bacon. The order in which we eat these is important, however. How many possibilities are there?

Here we are permuting P,P,P,P,M,M,M,B.

Thus $n = 8$, $n_1 = 4$, $n_2 = 3$ and $n_3 = 1$.

The total number of possibilities is then:

$$\frac{8!}{4!3!1!} = ... = 280$$

3.4 Sneaky Examples

Example 3.3. We are back at our local pizza shop the next day. Today, however, we're not sure. We could either have 4 plain, 3 mushroom, and 1 bacon (just like yesterday) or we could have 4 each of mushroom and bacon. The order in which we eat these is important, however. How many possibilities are there?

First we have to decide which option we're going with. There are two and they are disjoint so we will find the number of possibilities for each and add.

The first option is as the previous question, permuting P, P, P, P, M, M, M, B so there are 280 possibilities.

The second option is permuting M, M, M, M, B, B, B, B.

Thus $n = 8$, $n_1 = 4$, and $n_2 = 4$.

Thus for this option the total number of possibilities is then:

$$8!/(4!4!)$$

The total number of possibilities is then:

$$\frac{8!}{4!3!1!} + \frac{8!}{4!4!} = ... = 350$$

Example 3.4. Back at the pizza shop for the third day in a row again we want 4 plain, 3 mushroom, and 1 bacon but we don't want all 4 plain in a row. The order in which we eat these is important, however. How many possibilities are there now?

We know from earlier that there are 280 possibilitie permutations if we don't care about whether we have 4 plain in a row or not. Let's calculate the total number that would have 4 plain in a row and subtract.

This could happen 5 disjoint ways:

PPPP????, ?PPPP???, ??PPPP??, ???PPPP?, and ????PPPP

However each of these involves four unknowns and each of those is permuting M, M, M, B and so $n = 4$, $n_1 = 3$, and $n_2 = 1$, so each has $4!/(3!1!)$ possibilities and there are five of them.

Thus in total the number of possibilities is:

$$280 - 5\left(\frac{4!}{3!1!}\right) = ... = 240$$

4 Combinations of k Objects from n Categories

4.1 Introduction and Example

Suppose we want 7 sodas. The store has 3 types (and infinitely many of each). Since we're just putting them in a bag and taking them home, order does not matter. How many possibilities are there?

The approach to developing a formula for this is rather interesting. We imagine a soda as an X and a divider as a vertical line $|$. Because there are three types of sodas we'll use two vertical lines. So for example:

$XX|XXX|X$ means two sodas from the first type, three sodas from the second type, and one soda from the first type.

and

$XXXX|XXX|$ means four sodas from the first type, three sodas from the second type, and no sodas from the third type.

and

$||XXXXXXX$ means no sodas from the first type, no sodas from the second type, and seven sodas from the third type.

Think about this as a permutation of $7+2$ items of which 7 are in one category (the Xs) and 2 are in the second category (the $|$s).

We know that the number of ways to do this is:

$$\frac{(7+2)!}{7!2!}$$

Notice that this is equal to $C(7+3-1,7)$.

4.2 General Rule

Suppose we take a combination of k objects from n categories. The number of ways this can be done equals:

$$C(k+n-1,n-1)$$

This is also equal to:

$$C(k+n-1,k)$$

4.3 Basic Examples

Here are some more examples:

Example 4.1. We're having a party and need to buy paper hats. We need 20 paper hats and they come in four colors. How many possible ways could we buy the hats?

Here we are taking a combination of $k=20$ objects from $n=4$ categories so the total number of possibilities is:

$$C(20+4-1,20) = C(23,20) = \frac{20!}{(23-20)!20!} = ... = 171$$

4.4 Sneaky Examples

Here is one that is a bit more sneaky.

Example 4.2. Because of a terrible scheduling conflict we have two parties to organize in one day. For the first party we need to choose 20 paper hats from 3 different colors and for the first party we need to choose 10 paper hats from 4 different colors.

Here we are making both choices (two parties!). The first has $C(20+3-1,20)$ and the second has $C(10+4-1,10)$.

Thus the total number of possibilities is:

$$C(22,20)C(13,10) = \frac{22!}{(22-20)!20!} \cdot \frac{13!}{(13-10)!10!} = ... = 66066$$

5 Mix-N-Match of the Above

Here are some examples which will really require us to put everything together.

Example 5.1. We are trying to set up a panel discussion and we need 5 faculty members at a round table. There are 11 faculty members to choose from. Unfortunately two specific ones are annoying and if they're both at the table then they will insist that another specific third one is not. Absolute position doesn't matter but relative position does. How many possibilities are there?

First we ignore the rotations of the table and divide the problem into disjoint cases:

- The annoying two are both on the panel. In this case we need 3 more faculty members but we can't have the third one so we only have $11-3=8$ to choose from so $C(8,3)$ possibilities. Once we have chosen them we can permute them around the table 5! ways each so there are $C(8,3)5!$ possibilities.
- The first of the annoying ones is on the panel but the second is not. In this case we need 4 more faculty members but we can't have the second annoying one so we only have $11-2=9$ to choose from so $P(9,4)$ possibilities. Once we have chosen them we can permute them around the table 5! ways each so there are $C(9,4)5!$ possibilities.

- The second of the annoying ones is on the panel but the first is not. Again there are $P(9,4)5!$ possibilities.
- Neither of the annoying ones is on the panel. In this case we need 5 faculty members and we have 9 to choose from so $P(9,5)$ possibilities. Once we have chosen them we can permute them around the table 5! ways each so there are $C(9,5)5!$ possibilities.

Adding these together and then grouping them to account for rotations being identical the total number of possibilities is:

$$\frac{C(8,3)5! + C(9,4)5! + C(9,4)5! + C(9,5)5!}{5} = ... = 10416$$