CMSC 250: Propositional Logic

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1 Statements and Statement Variables

Definition 1.0.1. A *statement* (or *proposition*) is a sentence that is either true or false but not both.

Note 1.0.1. Note that the words *sentence*, *true*, and *false* are technically undefined here, and that's okay. For now we'll use an pretty obvious English language interpretation.

> **Example 1.1.** The sentence "$2 + 2 = 4$" is true and not false, hence it is a statement.

> **Example 1.2.** The sentence "$1 + 1 = 0$" is false and not true, hence it is a statement.

Note 1.0.2. You might argue there's some wiggle room here. If we are working in $\mathbb{Z}_2$ then "$1+1 = 0$" is true. So in the background here there's generally some well-understood context necessary.

Compare these two:

> **Example 1.3.** The sentence "$x > 0$" could be true or false, depending on the value of x. The implicit context here is that the value of x is not just unknown, but not fixed. It is therefore not a statement.

> **Example 1.4.** On the other hand consider the sentence "Let x be the amount of money in my wallet. Then $x > 20$." The value of x is fixed and hence this is a statement. The fact that you don't know if it's true or false doesn't matter, as it certainly is one or the other of those.

Note 1.0.3. Formally we might say that in the first case x is a variable but in the second case it's an (unknown) fixed constant.

Definition 1.0.2. A *statement variable* is a variable which represents an unknown statement. We'll use letters like p and q. We can then say, for example, that p will be true or false depending on what p represents.

2 Negation, Disjunction, Conjunction

There are three formal symbols we'll use to build more complicated statements out of simpler ones. Now we are talking about statements, so we're building things out of sentences which we know are either true or false.

2.1 Negation

Definition 2.1.1. If p is statement or statement variable then $\sim p$ is read "not p" and is the *negation* of p. It is defined according to:

$$\sim p = \begin{cases} \text{true} & \text{if } p \text{ is false} \\ \text{false} & \text{if } p \text{ is true} \end{cases}$$

Example 2.1. Suppose we have our statement:

$$p : 2 + 2 = 4$$

Then we know p is true and hence $\sim p$ is false.

Note 2.1.1. Arguably we have:

$$\sim p :\sim (2 + 2 = 4)$$

But we can easily see that we might prefer to write:

$$\sim p : 2 + 2 \neq 4$$

The fact that these are equivalent is obvious here, but might not be for more complicated examples.

Example 2.2. If p is a statement variable (representing an unknown statement) and I tell you that p is true, then you know that $\sim p$ is false.

2.2 Disjunctions

Definition 2.2.1. If p and q are statements or statement variables then $p \vee q$ is read "p or q" and is the *disjunction* of p and q. It is defined according to:

$$p \vee q = \begin{cases} \text{true} & \text{if one or both of } p \text{ and } q \text{ is true} \\ \text{false} & \text{if } p \text{ is false and } q \text{ is false} \end{cases}$$

Example 2.3. Suppose we have two statements:

$$p : 2 + 2 = 4$$
$$q : 4 + 5 = 1$$

Then we know p is true and q is false and hence $p \vee q$ is true.

Example 2.4. Suppose we have two statements:

$$p : 2 + 2 = 5$$
$$q : 4 + 5 = 1$$

Then we know p is false and q is false and hence $p \vee q$ is false.

Example 2.5. If p and a are statement variables (representing unknown statements) and I tell you that both are true, then you know that $p \vee q$ is true.

2.3 Conjunctions

Definition 2.3.1. If p and q are statements or statement variables then $p \wedge q$ is read "p and q" and is the *conjunction* of p and q. It is defined according to:

$$p \wedge q = \begin{cases} \text{true} & \text{if both of } p \text{ and } q \text{ are true} \\ \text{false} & \text{if one or both of } p \text{ and } q \text{ is false} \end{cases}$$

Example 2.6. Suppose we have two statements:

$$p : 2 + 2 = 4$$
$$q : 4 + 5 = 1$$

Then we know p is true and q is false and hence $p \wedge q$ is false.

Example 2.7. Suppose we have two statements:

$$p : 2 + 2 = 4$$
$$q : 4 + 5 = 9$$

Then we know p is true and q is true and hence $p \wedge q$ is true.

Example 2.8. If p and a are statement variables (representing unknown statements) and I tell you that p is false and q is true, then you know that $p \wedge q$ is false.

2.4 Xor

We use xor to mean one or the other but not both. We won't really use this much because it can be written in terms of $\sim$, $\vee$, and $\wedge$. However there are various ways to denote this including $p \oplus q$ and $p \veebar q$.

2.5 English Commentary

It's important to note that mathematics is precise but English is not. In English the word "or" is usually implied to be exclusive. For example when we say "I will eat chicken or steak." we usually mean one or the other but not both. In mathematics, however, "or" includes both.

There are also words in English which we might have to pause to translate. Two of these are:

$$p \text{ but } q \text{ means } p \text{ and } q$$

> **Example 2.9.** The sentence "I am tired but I will browse reddit some more" really means "I am tired and I will browse reddit some more".

And:

$$\text{Neither } p \text{ nor } q \text{ means } \sim p \text{ and } \sim q. \text{ Alternatively } (\sim p) \wedge (\sim p).$$

> **Example 2.10.** The sentence "It is neither snowing nor raining" really means "It is not snowing and it is not raining".

3 Statement Forms

3.1 Basic Definition

Definition 3.1.1. Now that we have our three basic building blocks, $\sim$, $\vee$, and $\wedge$, we can use these along with statement variables to build *statement forms.*

> **Example 3.1.** If p, q, and r are statement variables then $\sim (p \vee r) \wedge q$ is a statement form.

Note 3.1.1. Just to clarify an oftentimes confusing difference, from now on we'll use the term *statement form* to include single statement variables. If we want to refer specifically to the statement variables which make up a statement form we'll say *component statement variables.*

Thus p is a statement form, $p \wedge q$ is a statement form, and in the second case p and q are component statement variables within that statement form.

Whether a statement form is true or false depends on the truth of its component statement variables and can be quite hard to determine. In order to facilitate our understanding we'll (soon) define and draw truth tables to explore all possibilities.

3.2 Order of Operations

Negation takes precedence over conjunction and disjunction but neither conjuction nor disjunction takes precedence over the other, so parentheses are needed.

Example 3.2. $\sim p \vee q$ is unambiguous, it means the same as $(\sim p) \vee q$.

Example 3.3. $\sim (p \vee q)$ is different than the above.

Example 3.4. $p \vee q \wedge r$ is ambiguous. It's not clear which of $\vee$ or $\wedge$ is managed first. We need to write either $(p \vee q) \wedge r$ or $p \vee (q \wedge r)$ depending on which we mean, because they're different.

3.3 Truth Tables

Definition 3.3.1. A *truth table* is a table which starts by listing all possible combinations of T/F values for the component statement forms (or variables) and using these to determine the T/F values for the entire statement form. Usually, when necessary or just plain helpful, intermediate columns for intermediate statement forms are included.

Example 3.5. Here are truth tables for $\sim p$, $p \vee q$, and $p \wedge q$:

p	$\sim p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3.6. Here is a truth table for $\sim p \vee q$. It is helpful to have a column for $\sim p$.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth tables can get a bit out of control. If there are n statement variables then there will be 2^n rows since there will be 2^n possible combinations of T/F for the n statement variables.

Example 3.7. Here is a truth table for the statement form $(p \vee q) \wedge (\sim r)$. It is helpful to have a column for statement form $p \vee q$ and perhaps one for the statement form $\sim r$.

p	q	r	$p \vee q$	$\sim r$	$(p \vee q) \wedge (\sim r)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	F	T	F

4 Logical Equivalence

4.1 Definition

Definition 4.1.1. We say that two statement forms are logically equivalent, written using $\equiv$, when they have the same T/F value for every single possible T/F value combination of their component statement variables. A truth table can help us see if this is the case.

What follows are some classically important logical equivalences.

4.2 Double Negative

Example 4.1. Observe that $\sim(\sim p) \equiv p$. While this is intuitively obvious, here's a truth table:

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

We see that the columns for p and $\sim(\sim p)$ are exactly the same, which establishes the equivalency.

4.3 DeMorgan's Laws

Two very important logical equivalences are DeMorgan's Laws. Before giving the Laws, look at this example:

Example 4.2. Observe that $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$. Here's a truth table:

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$(\sim p) \wedge (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

We see that the columns for $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ are exactly the same, which establishes the equivalency.

Theorem 4.3.1. DeMorgan's Laws: If p and q are statements or statement forms then we have:

$$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$$
$$\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$$

Proof: The first is shown in the truth table above. You should try the second! $\mathcal{QED}$

4.4 Tautologies and Contradictions

Definition 4.4.1. A *tautology* is a statement form which is always true for all T/F values of its statement variables. If X is a tautology we'll just write $X \equiv t$. In a truth table we'll see a column full of T.

Example 4.3. The simplest of tautologies is $p \vee \sim p$.

Definition 4.4.2. A *contradiction* is a statement form which is always false for all T/F values of its statement variables. If X is a contradiction we'll just write $X \equiv c$. In a truth table we'll see a column full of F.

Example 4.4. The simplest of contradictions is $p \wedge \sim p$.

4.5 Summary of Common Logical Equivalences

Here are the most important logical equivalences. Some are obvious, some not so much. We have not proved most of them. You should try them using truth tables.

Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Laws	$p \wedge t \equiv p$	$p \vee c \equiv p$
Negation Laws	$p \vee \sim p \equiv t$	$p \wedge \sim p \equiv c$
Double Negative Law	$\sim (\sim p) \equiv p$	
Idempotent Laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound Laws	$p \vee t \equiv t$	$p \wedge c \equiv c$
DeMorgan's Laws	$\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$	$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$
Absorption Laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation of t and c	$\sim t \equiv c$	$\sim c \equiv t$

4.6 Working with Logical Equivalences

Once we have some logical equivalences we can use them to find others.

The first thing to note is that the component statement variables in the statement forms above are just that, variables, and so the above logical equivalences will still hold if the component statement variables are replaced by other statement forms.

Example 4.5. We have the identity law $p \wedge t \equiv p$. Think of the statement variable p as a placeholder, so think $X \wedge t \equiv X$ for any statement form X. Thus, for example, we get all of the following:

$$(\sim p) \wedge t \equiv \sim p$$

and

$$(p \vee q) \wedge t \equiv p \vee q$$

and

$$(\sim p \wedge q) \wedge t \equiv \sim p \wedge q$$

In addition we can build new laws.

Example 4.6. Let's show that $\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p$.

$$\begin{aligned}
\sim (\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim (\sim p) \vee \sim q) \wedge (p \vee q) && \text{DeMorgan's Law} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{Double Negative Law} \\
&\equiv p \vee (\sim q \wedge q) && \text{Distributive Law (backwards)} \\
&\equiv p \vee c && \text{Negation Law} \\
&\equiv p && \text{Identity Law}
\end{aligned}$$

5 Conditional Statements

5.1 Introduction and Definition

In mathematics statements and statement forms often contain implications:

If p then q.

Definition 5.1.1. If p and q are statements or statement forms then we can construct the *conditional* statement or statement form $p \to q$ (also called the *implication*) read as "if p then q" or "p implies q". Here p is the *hypothesis* or *antecedent* and q is the *conclusion* or *consequent*.

We're first interested in whether this statement is true or not. Keep in mind that we're not interested in whether p is true or q is true but rather whether $p \to q$ is true.

In order to understand whether this statement is true or not it is critically important (to help avoid confusion) to understand that the nature of "If p then q" is not really causal, even though it often can be thought of that way.

In other words saying that $p \to q$ is true does not mean that p being true must cause q to be true but rather that whenever p is true then q is also true.

Now let's consider some situations for the statement form $p \to q$.

1. First, consider this example:

   > **Example 5.1.** Let's put:
   >
   > $$\begin{aligned} p &: \text{I open my eyes between 1pm and 2pm} \\ q &: \text{It will be daytime} \end{aligned}$$
   >
   > Then $p \to q$ becomes:
   >
   > If I open my eyes between 1pm and 2pm then it will be daytime.
   >
   > This example is clearly true. Note that from a causal perspective opening your eyes between 1pm and 2pm doesn't make it daytime.

   We see here that whenever p is true we find that q is also true then it seems reasonable to say that $p \to q$ is true.

   Two other examples which suggest the same:

   > **Example 5.2.** Let's put:
   >
   > $$\begin{aligned} p &: \text{I arrive at Union Station at 9am for the 9am train to NYC} \\ q &: \text{The train will be there} \end{aligned}$$
   >
   > Then $p \to q$ becomes:
   >
   > If I arrive at Union Station at 9am for the 9am train to NYC then the train will be there.
   >
   > Also true. Note again that from a causal perspective I didn't cause the train to be there.

   > **Example 5.3.** Let's put:
   >
   > $$\begin{aligned} p &: x > 3 \\ q &: x + 2 > 5 \end{aligned}$$
   >
   > In this case we'll say that x is an unknown fixed real number. Then $p \to q$ becomes:
   >
   > If $x > 3$ then $x + 2 > 5$.

Again, also true. In this example we might think that the relationship is arguably causal but really it's not. This example is simply saying that whenever $x > 3$ then it will also be the case that $x + 2 > 5$. Imagine infinitely many x's lying around. If you pick one up and it's greater than 3, then $x + 2$ will be greater than 5.

2. Second, consider this example:

   Example 5.4. Let's put:

   $$p : \text{My first name is Justin}$$
   $$q : \text{My last name is Time}$$

   Then $p \to q$ becomes:

   If my first name is Justin then my last name is Time.

   This is clearly false. My first name is Justin but my last name isn't Time.

   We see here that when p is true and q is false it seems reasonable to say that $p \to q$ is false.

3. Third, consider these examples:

   Example 5.5. Let's put:

   $$p : x \text{ is a real number with } x^2 < 0$$
   $$q : \text{I am a muppet}$$

   Then $p \to q$ becomes:

   If x is a real number with $x^2 < 0$ then I am a muppet.

   Example 5.6. Let's put:

   $$p : x \text{ is a real number with } x^2 < 0$$
   $$q : \text{I am a human}$$

   Then $p \to q$ becomes:

   If x is a real number with $x^2 < 0$ then I am a human.

   Hmmm. Are these statements true? The hypotheses are not true. So what should we say about the conditionals?

   The way we address this is to understand that $p \to q$ can only be said to be false if p is true and q is false. When p itself is false we say that $p \to q$ is true. It's a special kind of true, called *vacuously true*.

To summarize, $p \to q$ obeys the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

5.2 Rewriting using Not and Or

The truth table above lets us also understand the following logical equivalence:

$$p \to q \equiv \sim p \vee q$$

Stop for a minute and consider that this makes sense if we think about the condition under which each is false:

> $p \to q$ is false when p is true and q is false.
>
> $\sim p \vee q$ is false when $\sim p$ is false (p is true) and q is false.

5.3 Negating a Conditional

One reason for recognizing that $p \to q \equiv \sim p \vee q$ is that it allows us to negate:

$$\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$$

Again, this makes sense, as it is not the case that $p \to q$ precisely when p is true and q is false.

Please keep in mind that when the conclusion it itself a statement form that we may wish to go further. In the following two examples we're not saying that you must go further but rather that the logical equivalences all hold.

Example 5.7. Consider the negation of $p \to (q \vee \sim r)$. We use one of DeMorgan's Laws as well:

$$\begin{aligned} \sim (p \to (q \vee \sim r)) &\equiv p \wedge \sim (q \vee \sim r) \\ &\equiv p \wedge (\sim q \wedge r) \end{aligned}$$

Example 5.8. Consider the negation of $p \to (q \to r)$ in which we can go further still.

$$\begin{aligned}\sim(p \to (q \to r)) &\equiv p\wedge \sim(q \to r)\\ &\equiv p \wedge (q\wedge \sim r)\\ &\equiv (p \wedge q)\wedge \sim r\\ &\equiv\sim((p \wedge q) \to r)\end{aligned}$$

5.4 Converse

Most theorems in mathematics are conditionals. There are one or more hypothesis which are assumed and generally one conclusion which we wish to prove. For most theorems we might also be interested in whether the "reverse" is true.

Definition 5.4.1. The *converse* of $p \to q$ is the conditional $q \to p$.

Example 5.9. The converse of "If it is raining then it is cloudy" is "If it is cloudy then it is raining".

Note 5.4.1. These converse and the original are not logically equivalent.

5.5 Contrapositive

Definition 5.5.1. The *contrapositive* of $p \to q$ is the conditional $\sim q \to\sim p$.

Example 5.10. The contrapositive of "If it is raining then it is cloudy" is "If it is not cloudy then it is not raining".

Note 5.5.1. The contrapositive is in fact logically equivalent to the original conditional. In other words $p \to q \equiv\sim q \to\sim p$.

5.6 Inverse of a Conditional

Definition 5.6.1. The *inverse* of $p \to q$ is the conditional $\sim p \to\sim q$.

Example 5.11. The inverse of "If it is raining then it is cloudy" is "If it is not raining then it is not cloudy".

Note 5.6.1. The inverse is not logically equivalent to the original but is equivalent to the converse.

5.7 A Fun Diagram

In reality since we can take converses, contrapositives, and inverses of any conditional we can see the process in the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
q \to p \arrow[r, leftrightarrow, "\text{contrapositive}", "\equiv"'] \arrow[d, leftrightarrow, "\text{converse}"', "\not\equiv"] & \sim p \to \sim q \arrow[d, leftrightarrow, "\not\equiv"', "\text{converse}"] \arrow[dl, leftrightarrow, "\not\equiv"', "\text{inverse}"] \\
p \to q \arrow[r, leftrightarrow, "\equiv", "\text{contrapositive}"'] & \sim q \to \sim p \arrow[ul, leftrightarrow, bend right=100, "\not\equiv", "\text{inverse}"']
\end{tikzcd}$$

5.8 Biconditionals

Definition 5.8.1. If p and q are statments or statement forms then we can construct the *biconditional* statement or statement form $p \leftrightarrow q$, read as "p if and only q" and often written as "p iff q". The biconditional is the conjunction of $p \to q$ and its converse $q \to p$. In other words:

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Note 5.8.1. There are other logically equivalent ways to think about the biconditional, for example:

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$$

This should make intuitive sense; either they're both true or they're both false, bit this can also be proved using logical equivalences that we already know:

$$\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) && \text{Definition} \\
&\equiv (\sim p \vee q) \wedge (\sim q \vee p) && \text{Logical Equivalence} \\
&\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) && \text{Distributive Laws} \\
&\equiv ((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p)) && \text{Distributive Laws} \\
&\equiv ((\sim p \wedge \sim q) \vee c) \vee (c \vee (q \wedge p)) && \text{Negation Laws} \\
&\equiv (\sim p \wedge \sim q) \vee (q \wedge p) && \text{Identity Laws}
\end{aligned}$$

5.9 Necessary and Sufficient

The terms "necessary" and "sufficient" in English correspond to conditionals in mathematics.

Consider the statement:

For me to pass the class it is sufficient to have a 60% or more.

Note that this is the same as:

It is sufficient for me to get a 60% or more to pass the class.

These statements basically mean:

If I get a 60% or more then I will pass the class.

Thus we see:

For p it is sufficient to have q (aka q is sufficient for p) means $q \to p$

On the other hand consider the statement:

To make an apple pie it is necessary to buy apples.

Note that this is the same as:

It is necessary for me to buy apples to make an apple pie.

These statements basically mean:

If I make an apple pie then I buy apples.

Thus we see:

For p it is necessary to have q (aka q is necessary for p) means $p \to q$.

6 Arguments and Argument Forms

6.1 Arguments

Suppose I make the following argument about someone named Frank. You don't know Frank, or anything about him:

	Frank is over 40 years old.
	If Frank is over 40 years old then Frank is happy.
Therefore:	Frank is happy

You would probably agree that this argument is valid. Let's ponder what we mean by this for a minute. You might stop and ask "Is Frank actually over 40 years old? Maybe not!" but would that affect whether the argument is valid? Well, what might "valid mean?

Suppose we identified the first two statements as hypotheses and the last one as a conclusion:

	Frank is over 40 years old.	Hypothesis
	If Frank is over 40 years old then Frank is happy.	Hypothesis
Therefore:	Frank is happy	Conclusion

We might suggest that this argument is valid not because it's all true but because *if* the hypotheses are both true, *then* we can conclude for sure that the conclusion is also true.

On the other hand consider this argument:

	Frank is over 40 years old.	Hypothesis
	If Frank owns a motorcycle then Frank is happy.	Hypothesis
Therefore:	Frank is happy	Conclusion

We might suggest that this argument is not valid because even if the two hypotheses are true we cannot conclude for certain that the conclusion is true.

We make these sorts of arguments all the time and the goal of this section is to formalize them.

6.2 Argument Forms

Definition 6.2.1. An *argument form* consists of a collection of statement forms, called *hypotheses*, and an additional single statement form, called a *conclusion*

We write an argument form with one statement form per line and with the symbol $\therefore$ (read "therefore") before the final statement, which is the conclusion.

Example 6.1. The following is an argument form with two hypotheses (and one conclusion). You might recognize that it's a general form of the opening argument from the section:

	p	Hypothesis
	$p \to q$	Hypothesis
$\therefore$	q	Conclusion

Example 6.2. The following is another one, not as familiar:

$$\begin{array}{lll} & q & \text{Hypothesis} \\ & p \vee q & \text{Hypothesis} \\ \therefore & p \to q & \text{Conclusion} \end{array}$$

6.3 Valid and Invalid Arguments

Informally, we are interested in whether the hypotheses together imply the conclusion.

Example 6.3. In this example again:

$$\begin{array}{lll} & q & \text{Hypothesis} \\ & p \vee q & \text{Hypothesis} \\ \therefore & p \to q & \text{Conclusion} \end{array}$$

We are interested in whether this statement is true:

$$\Big[q \wedge (p \vee q)\Big] \to \Big[p \to q\Big]$$

More generally if the hypotheses are hyp_1, hyp_2, ... , hyp_n and the conclusion is *conc*, we are interested in whether this statement is true:

$$\Big[hyp_1 \wedge hyp_2 \wedge ... \wedge hyp_n\Big] \to \big[conc\big]$$

We know that this implication is (vacuously) true when $hyp_1 \wedge hyp_2 \wedge ... \wedge hyp_n$ is false, meaning when one or more of the hyp_i is false, and so we only need to focus on the following situation:

When all of the hypotheses hyp_1, hyp_2, ... , hyp_n are true, is the conclusion *conc* true, too?

$$\underbrace{\Big[hyp_1 \wedge hyp_2 \wedge ... \wedge hyp_n\Big]}_{\text{When these are all true...}} \to \underbrace{\big[conc\big]}_{\text{...is this true?}}$$

This leads to the following definition:

Definition 6.3.1. An argument form is *valid* if, whenever the hypotheses are all true, the conclusion must also be true. An argument form is *invalid* if there are one or more situations in which the hypotheses are all true but the conclusion is false.

One way to determine whether an argument form is valid is to write out a truth table which has one row for each combination of the component statement variables. We then inspect the rows in which the hypotheses are all true. If, for each of those rows, the conclusion is true, then the argument is valid. On the other hand if there are one or more rows in which the hypotheses ar all true but the conclusion is false then the argument is invalid.

Example 6.4. Consider the following argument form again:

$$\begin{array}{ll} & q \\ & p \vee q \\ \therefore & p \to q \end{array}$$

Here is a truth table for all possible p and q which also displays $p \vee q$ and $p \to q$:

p	q	$p \vee q$	$p \to q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

Now examine all the rows in which both q and $p \vee q$ are true:

p	q	$p \vee q$	$p \to q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

Observe that in both of these rows, $p \to q$ is also true. Thus the argument is valid.

On the other hand:

Example 6.5. Consider the following argument form:

$$\begin{array}{ll} & p \\ & p \vee q \\ \therefore & p \to q \end{array}$$

This has the same truth table as the previous example. Let's examine all the rows in which both q and $p \vee q$ are true

p	q	$p \vee q$	$p \to q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

Observe that in one of these rows, $p \to q$ is false. Thus the argument is invalid.

6.4 Modus Ponens and Modus Tollens

The two most common valid argument forms are modus ponens and modus tollens.

Definition 6.4.1. The argument form *modus ponens* is:

$$\begin{array}{ll} & p \to q \\ & p \\ \therefore & q \end{array}$$

Stop and think about this and it makes perfect sense. If you know that $p \to q$ and p, then you can conclude q.

Similarly:

Definition 6.4.2. The argument form *modus tollens* is:

$$\begin{array}{ll} & p \to q \\ & \sim q \\ \therefore & \sim p \end{array}$$

Again, clearly if you know $p \to q$ and $\sim q$, then you can conclude $\sim p$.

6.5 Rules of Inference

Definition 6.5.1. A *rule of inference* is a valid argument form.

For example, modus ponens and modus tollens are rules of inference. Here are the most common rules of inference:

Modus Ponens	Modus Tollens	Generalization
$p \to q$ p $\therefore\ q$	$p \to q$ $\sim q$ $\therefore\ \sim p$	p $\therefore\ p \vee q$
Specialization	**Conjunction**	**Elimination**
$p \wedge q$ $\therefore\ p$	p q $\therefore\ p \wedge q$	$p \vee q$ $\sim p$ $\therefore\ q$
Transitivity	**Cases**	**Contradiction**
$p \to q$ $q \to r$ $\therefore\ p \to r$	$p \vee q$ $p \implies r$ $q \implies r$ $\therefore\ r$	$\sim p \to c$ $\therefore\ p$

6.6 Using Rules of Inference

We can use rules of inference to build new rules, and on and on and on, thereby showing that an argument form is valid. We do this by using the hypotheses along with known rules of inference to contruct new hypotheses, and we continue this until we get our conclusion.

Keep in mind with these rules of inference that p, q, etc. can be not just statement variables but statement forms.

Example 6.6. Here is a really simple argument form:

$$\begin{array}{rl} & p \\ & p \to q \\ & q \to r \\ \therefore & r \end{array}$$

Before being very organized just think about this intuitively. We have three hypotheses: p, $p \to q$ and $q \to r$. Suppose that all three of these are true. Since p is true and $p \to q$ is true, this means q must be true, too, by modus ponens. This is a new hypothesis we can add to the mix. Then since q is true and $q \to r$ is true, this means r is true, too, by modus ponens again. Then we have our conclusion.

This can be approached in a more organized manner. First we write down the hypotheses we're given, and number them:

$$\begin{array}{ll} p & (1) \\ p \to q & (2) \\ q \to r & (3) \end{array}$$

Then we observe that modus ponens applies to (1) and (2) to yield the conclusion q so we add this to our list:

$$\begin{array}{ll} p & (1) \\ p \to q & (2) \\ q \to r & (3) \\ q & (4) \text{ Modus Ponens using (1) and (2)} \end{array}$$

Then we observe that modus ponens applies to (4) and (3) to yield the conclusion r so we add this to our list:

p	(1)
$p \to q$	(2)
$q \to r$	(3)
q	(4) Modus Ponens using (1) and (2)
r	(5) Modus Ponens using (4) and (3)

Thus we have reached r and the conlusion holds, thus the argument form is valid.

Of course we don't need to rewrite the entire list each time, we simply build onto it as we go.

Here is another example:

Example 6.7. Consider the following argument form:

$$\begin{array}{ll} & p \vee q \\ & q \to r \\ & p \wedge s \to t \\ & \sim r \\ & \sim q \to u \wedge s \\ \therefore & t \end{array}$$

To see that this is valid, we start with the hypotheses and build from there:

$p \vee q$	(1)
$q \to r$	(2)
$p \wedge s \to t$	(3)
$\sim r$	(4)
$\sim q \to u \wedge s$	(5)
$\sim q$	(6) Modus Tollens using (2) and (4)
p	(7) Eliminiation using (1) and (6)
$u \wedge s$	(8) Modus Ponens using (5) and (6)
s	(9) Specialization using (8)
$p \wedge s$	(10) Conjunction using (7) and (9)
t	(11) Modus Ponens using (10) and (3)

Note 6.6.1. It is important to note that there is often more than one way to build an argument.