

CMSC 250: Sequences, Sums, and Products

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1 Sequences

1.1 Intuition and Formal Definition

Sequences of numbers arise all over the place in both mathematics and computer science, in both limits and for loops.

Informally a sequence is just an infinite list of numbers, one after another.

Example 1.1. Here are some examples:

1, 2, 3, 4, ...	Looks like it increases by 1 each time!
42, 43, 44, 45, ...	Same as above but starts higher!
0, 6, 23, -8.2 , π , e^2 , ...	Pattern Not Clear!
0.1, 0.01, 0.001, 0.0001, ...	Looks like it divides by 10 each time!

There's nothing wrong with listing the elements of the sequence when the pattern is clear. However sometimes it isn't.

There is a formal definition, however.

Definition 1.1.1. A *sequence* is a function f whose domain is a set of the form $D = \{n_0, n_0 + 1, n_0 + 2, \dots\}$ where n_0 is a nonnegative integer and which outputs real numbers. The domain is called the set of *indices* and n_0 is the *starting index*.

Example 1.2. If $D = \{3, 4, 5, 6, \dots\}$ and $f(n) = n^2$ then we get the sequence:

$$f(3), f(4), f(5), \dots = 3^2, 4^2, 5^2, \dots$$

1.2 Traditional Notation

It's not very traditional to denote sequences the formal way and instead of $f(n)$ we usually write a_n and then simply give the starting index.

Example 1.3. The sequence above would traditionally be defined by:

$$a_n = n^2 \text{ for } n \geq 3$$

Then we would have $a_3 = 3^2 = 9$, $a_4 = 4^2 = 16$, and so on.

Alternately we can also use curly bracket notation.

Example 1.4. The sequence above can also be defined by:

$$\{n^2\}_{n=3}$$

The only downside to this notation is that it looks vaguely set-like. I'll avoid it here in the notes for this reason.

1.3 Recursive Sequences

Another common way to define a sequence is recursively. A recursive definition involves giving one or more starting terms and then a formula for creating successive terms from previous terms.

Example 1.5. We may define a sequence as follows:

$$\begin{aligned}a_1 &= 4 \\ a_k &= 2a_{k-1} + 1 \text{ for } k \geq 2\end{aligned}$$

We can then calculate each term in turn:

$$\begin{aligned}a_2 &= 2a_1 + 1 = 2(4) + 1 = 9 \\ a_3 &= 2a_2 + 1 = 2(9) + 1 = 19 \\ &\vdots = \vdots\end{aligned}$$

We can also start by giving more than just the first term.

Example 1.6. We may define a sequence as follows:

$$\begin{aligned}a_1 &= 4 \\ a_2 &= -1 \\ a_k &= a_{k-1}^2 - a_{k-2} \text{ for } k \geq 3\end{aligned}$$

We can then calculate each term in turn:

$$\begin{aligned}a_3 &= a_2^2 - a_1 = 1 - 4 = -3 \\ a_4 &= a_3^2 - a_2 = 9 - (-1) = 10 \\ &\vdots = \vdots\end{aligned}$$

1.4 Conversion Between Definitions

In general it is hard to take a recursively defined sequence and give an a_n formula for it and it is also hard to do the reverse.

However in some cases we can. There are some algebraically technical approaches to this but we just want to be able to see the pattern in some examples.

Example 1.7. The sequence $a_i = 3i$ for $i \geq 1$ has terms 3, 6, 9, 12, ... and can also be defined recursively by $a_0 = 3$ and $a_k = 3 + a_{k-1}$ for $k \geq 1$.

Example 1.8. The sequence $a_i = 3^i$ for $i \geq 1$ has terms 3, 9, 27, 81, ... and can also be defined recursively by $a_0 = 3$ and $a_k = 3a_{k-1}$ for $k \geq 1$.

1.5 Shifting Starting Indices

It may be helpful (as we will see) to shift the indices of a sequence so that it starts at a different index.

Example 1.9. Suppose $a_k = k^2$ for $k \geq 4$. Suppose we wanted to have $k \geq 0$ instead. What could we do to the a_k ? Well, it's not hard to see that we would need to have $a_k = (k+4)^2$ so that the first number we're squaring is still 4

There is a way to be formulaic about this. Let's assume that k is the sequence variable.

Theorem 1.5.1. To change the starting index from a to b , replace all the k by $k + (a - b)$.

Example 1.10. Suppose $a_k = 5^k + k - 7$ for $k \geq 4$. To change the starting index to 1 we replace k by $k + (4 - 1) = k + 3$ to get $a_k = 5^{k+3} + (k+3) - 7 = 5^{k+3} + k + 10$.

2 Sums

2.1 Notation

Suppose we have a sequence a_n and wish to add up some finite number of terms, for example:

We use summation notation:

$$\sum_{n=\text{Starting Index}}^{\text{Ending Index}} a_n$$

Example 2.1. Here are some examples:

$$\sum_{n=2}^{10} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10}$$
$$\sum_{i=3}^{42} n^2 + n = (3^2 + 3) + (4^2 + 4) + \dots + (42^2 + 42)$$

2.2 Evaluation - Just Do It!

In simple cases if we wish to evaluate a sum we can simply add up the numbers.

Example 2.2. For example:

$$\sum_{n=2}^5 n^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

2.3 Special Case Formulas

We have several common sums which occur frequently:

$$\sum_{k=1}^n 1 = 1 + 1 + \dots + 1 = n$$

$$\sum_{k=1}^n n = 1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \text{Gauss' Sum}$$

$$\sum_{k=1}^n n^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{Sum of Squares}$$

$$\sum_{k=0}^n r^k = 1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r} \quad \text{Geometric Sum}$$

There are various ways to prove these but for now we'll wait until we have mathematical induction.

However it is worth noting that these formulas can be used to figure out more complicated sums.

Example 2.3. Consider the sum:

$$\sum_{k=0}^{50} 3(2)^{4k+1}$$

This looks a bit like the Geometric Sum but in order to use the formula we need to do some rewriting:

$$\begin{aligned} \sum_{k=0}^{50} 3(2)^{4k+1} &= 3 \sum_{k=0}^{50} (2)^{4k+1} \\ &= 3 \sum_{k=0}^{50} 2^1 (2^4)^k \\ &= 6 \sum_{k=0}^{50} (16)^k \\ &= 6 \left(\frac{1 - 16^{51}}{1 - 16} \right) \end{aligned}$$

Here is a more complex example:

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Example 2.4. Consider the sum:

$$\sum_{k=2}^{20} 1 + k + 5(0.3)^k$$

First note we can split it up and factor out the 5:

$$\sum_{k=2}^{20} 1 + \sum_{k=2}^{20} k + 5 \sum_{k=2}^{20} (0.3)^k$$

Each of these is familiar but the starting indices are not quite right. For the first, it's easy to calculate anyway:

$$\sum_{k=2}^{20} 1 = 19$$

For the second two we can change the starting indices as long as we subtract the parts we're adding:

$$\sum_{k=2}^{20} k = \left[\sum_{k=1}^{20} k \right] - 1 = \frac{20(20+1)}{2} - 1$$

and:

$$\sum_{k=2}^{20} (0.3)^k = \left[\sum_{k=0}^{20} (0.3)^k \right] - 1 - 0.3 = \frac{1 - (0.3)^{21}}{1 - 0.3} - (0.3)^0 - (0.3)^1$$

All together:

$$\sum_{k=2}^{20} 1 + k + 5(0.3)^k = 19 + \frac{20(20+1)}{2} - 1 + 5 \left(\frac{1 - (0.3)^{21}}{1 - 0.3} - 1 - 0.3 \right)$$

Here is a nested example:

Example 2.5. Consider the sum:

$$\sum_{n=1}^{10} \sum_{i=1}^n 1$$

We evaluate this from the inside out. Parentheses may help:

$$\sum_{n=1}^{10} \left[\sum_{i=1}^n 1 \right] = \sum_{n=1}^{10} [n] = \frac{10(10+1)}{2}$$

Here is a more complicated nested example:

Example 2.6. Consider the sum:

$$\sum_{n=5}^{50} \sum_{i=1}^{n+1} i$$

We evaluate this from the inside out. Parentheses may help:

$$\begin{aligned} \sum_{n=5}^{50} \left[\sum_{i=1}^{n+1} i \right] &= \sum_{n=5}^{50} \left[\frac{(n+1)(n+1+1)}{2} \right] \\ &= \sum_{n=5}^{50} \frac{1}{2} [n^2 + 3n + 2] \\ &= \frac{1}{2} \left[\sum_{n=5}^{50} n^2 + 3 \sum_{n=5}^{50} n + 2 \sum_{n=5}^{50} 1 \right] \end{aligned}$$

At this point we might want to do the remaining sums individually:

$$\begin{aligned} \sum_{n=5}^{50} n^2 &= \left[\sum_{n=1}^{50} n^2 \right] - 1^2 - 2^2 - 3^2 - 4^2 = \frac{50(50+1)(2(50)+1)}{6} - 30 \\ \sum_{n=5}^{50} n &= \left[\sum_{n=1}^{50} n \right] - 1 - 2 - 3 - 4 - 5 = \frac{50(50+1)}{2} - 15 \\ \sum_{n=5}^{50} 1 &= 51 \end{aligned}$$

Thus together the answer is:

$$\frac{1}{2} \left[\frac{50(50+1)(2(50)+1)}{6} - 30 + 3 \left(\frac{50(50+1)}{2} - 15 \right) + 2(51) \right]$$

2.4 Telescoping Sums

Sometimes we find that when we write out a sum almost all the terms will cancel. These are known as telescoping sums.

Here is an example:

Example 2.7. Consider the sum:

$$\sum_{i=1}^{100} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

If we write out a number of the terms in this sum:

$$\sum_{i=1}^{100} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{99} - \frac{1}{100} \right) + \left(\frac{1}{100} - \frac{1}{101} \right)$$

We see that all but the first and last fractions cancel, leaving a result of:

$$\frac{1}{1} - \frac{1}{101}$$

3 Products

3.1 Notation

Suppose we have a sequence a_n and wish to multiply some finite number of terms, for example:

We use product notation:

$$\prod_{n=\text{Starting Index}}^{\text{Ending Index}} a_n$$

Example 3.1. Here is some notation and what it means:

$$\prod_{n=2}^{10} \frac{1}{n} = \left(\frac{1}{2}\right) \left(\frac{1}{3}\right) \dots \left(\frac{1}{10}\right)$$

$$\prod_{i=3}^{42} n^2 + n = (3^2 + 3)(4^2 + 4) \dots (42^2 + 42)$$

3.2 Evaluation - Just Do It!

In simple cases if we wish to evaluate a product we can simply multiply the numbers.

Example 3.2. For example:

$$\prod_{n=2}^5 n^2 = (2^2)(3^2)(4^2)(5^2) = 14400$$

3.3 Telescoping Products

Sometimes we find that when we write out a product almost all the terms will cancel. These are known as telescoping products.

Here is an example:

Example 3.3. Consider the product:

$$\prod_{i=5}^{100} \frac{i}{i+1}$$

If we write out a number of the terms in this sum:

$$\prod_{i=5}^{100} \frac{i}{i+1} = \left(\frac{5}{6}\right) \left(\frac{6}{7}\right) \left(\frac{7}{8}\right) + \dots + \left(\frac{99}{100}\right) \left(\frac{100}{101}\right)$$

We see that most of it cancels, leaving a result of:

$$\frac{5}{101}$$