CMSC 250: Set Theory and Proofs

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1 Sets and Notation

1.1 Basic Definitions

Definition 1.1.1. A *set* is a collection of (perhaps no) objects, or *elements*. What those elements are is entirely irrelevant to the definition. Think of a set as a box which contains (perhaps no) things.

$\square$

There is no repetition in a set, meaning each element must be unique. You could, for example, have variations on an element, such as a regular number 4 and a boldface number **4**.

There is no order in a set; in other words order does not matter.

1.2 Set Notation Types

There are several ways to denote the elements in a set. Some sets have specific notations because they arise so frequently. We have seen $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$, $2\mathbb{Z}$ and so on.

Definition 1.2.1. We say that *x is an element of/in a set A*, written $x \in A$, if it belongs in the set. We write $x \notin A$ if not.

$\square$

Example 1.1. For example $3 \in \mathbb{Z}$ and $3.4 \notin \mathbb{Z}$.

$\square$

We can also explicitly list the elements in a set using {} notation.

Example 1.2. Here are a few:

- $\{1, 2, 3, 4\}$ contains four things, the numbers 1,2,3, and 4.
- {primes} contains all the primes.

$\square$

Definition 1.2.2. The *empty set* is denoted by {} or $\varnothing$ and contains nothing. It is an empty box.

$\square$

Note 1.2.1. Note that $\{\varnothing\}$ is not the empty set, rater it is a set containing one thing, that thing is the empty set! Think of it as a box containing an empty box.

$\square$

We also have notation for sets which comes from calculus using interval notation.

Example 1.3. The set $[1, 2]$ contains all real number from 1 to 2 inclusive. Note that this is very different than the set $\{1, 2\}$ which contains just the integers 1 and 2.

□

Lastly we can use set-builder notation to build sets:

Definition 1.2.3. A set is defined using *set-builder notation* using the notation:

$$\{\text{element(s)} \,|\, \text{conditions on element(s)}\}$$

The vertical bar is usually read as "such that".

□

Example 1.4. Here are some examples:

- $\{x \,|\, x \in \mathbb{Z} \wedge 3 \mid x\}$ is the set of integers which are multiples of 3.
- $\{x \,|\, x \in \{\text{people}\} \wedge x \text{ is over 6ft tall}\}$ is the set of people over 6 feet tall.
- $\{a \,|\, a \in \mathbb{Z} \wedge ((a \equiv 1 \mod 10) \vee (a \equiv 3 \mod 10))\}$

□

It's not infrequent that when the elements are taken from a set that this fact is made clear before the vertical bar:

Example 1.5. For example:

$$\{x \,|\, x \in \mathbb{Z} \wedge 3 \mid x\} = \{x \in \mathbb{Z} \,|\, (3 \mid x)\}$$

□

Also it's not infrequent that an expression is used on the left.

Example 1.6. For example the odd numbers can be written as $\{2k+1 \,|\, k \in \mathbb{Z}\}$.

□

2 Comparison

2.1 Subsets and Proper Subsets

Definition 2.1.1. Given sets A and B we say that A *is a subset of* B, written $A \subseteq B$, if every element in A is also in B. In other words:

$$A \subseteq B \longleftrightarrow \forall x, (x \in A \to x \in B)$$

□

Example 2.1. For example $\{1, 2, 3, 4\} \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{R}$.

□

Definition 2.1.2. Given sets A and B we say that A *is a proper subset of B*, written $A \subset B$ or $A \subsetneq B$, if every element in A is also in B but B has more. In other words:

$$A \subsetneq B \longleftrightarrow (\forall x, (x \in A \rightarrow x \in B) \wedge (\exists y \in B, y \notin A))$$

□

Example 2.2. For example $\{1, 2, 3, 4\} \subsetneq \mathbb{Z}$ and $\mathbb{Z} \subsetneq \mathbb{R}$.

□

Note 2.1.1. Observe that:

$$A = B \longleftrightarrow (A \subseteq B \wedge B \subseteq A)$$

□

2.2 Equality

Definition 2.2.1. We say that two sets A and B are *equal* and write $A = B$ if they have exactly the same elements. More specifically $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

□

Example 2.3. For example $\{x \in \mathbb{Z} \,|\, (3|x)\} = 3\mathbb{Z}$.

□

3 Combining

3.1 Union

Definition 3.1.1. Given sets A and B we define *the union of A and B*, denoted $A \cup B$ to be the set of elements with are in A or in B or in both. In other words:

$$A \cup B = \{x \,|\, x \in A \vee x \in B\}$$

Thus:

$$x \in A \cup B \longleftrightarrow x \in A \vee x \in B$$

□

Example 3.1. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

□

3.2 Intersection and Disjoint Sets

Definition 3.2.1. Given sets A and B we define *the intersection of A and B*, denoted $A \cap B$ to be the set of elements with are in both A and B. In other words:

$$A \cap B = \{x \,|\, x \in A \land x \in B\}$$

Thus:

$$x \in A \cap B \longleftrightarrow x \in A \land x \in B$$

□

Example 3.2. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$ then $A \cap B = \{4, 5\}$.

□

Definition 3.2.2. We say sets are *disjoint* if $A \cap B = \varnothing$. This means they have nothing in common!

□

Example 3.3. The sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$ are not disjoint but the sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ are.

□

3.3 Complements

In order to define the notion of things "not in" a set we have to have a sense of a universal set.

Definition 3.3.1. In a given problem or situation the *universal set*, denoted U, generally denotes a large set for which everything else is taken to be a subset.

□

Example 3.4. When working with sets of integers we would take $U = \mathbb{Z}$ whereas when working with sets of reals we would take $U = \mathbb{R}$.

$\square$

Definition 3.3.2. Suppose $A \subseteq U$. Then we define *the complement of A*, denoted A^c or A', to be the set of things in U but not in A. That is:

$$A^c = A' = \{x \in U \mid x \notin A\}$$

Thus:

$$x \in A^c \longleftrightarrow x \notin A$$

$\square$

Example 3.5. If $U = \mathbb{Z}$ and $A = 2\mathbb{Z}$ then $A^c = \{2k + 1 \mid k \in \mathbb{Z}\}$.

$\square$

3.4 Subtraction

Lastly we define set subtraction.

Definition 3.4.1. If A and B are sets then we define the set $A - B$ to be the set of things in A but not in B. Note that B doesn't need to be a subset of A for this to make sense. That is:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

The notation $A \setminus B$ is also used but is going out of fashion.

Thus:

$$x \in A - B \longleftrightarrow (x \in A) \wedge (x \notin B)$$

$\square$

Example 3.6. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$ then $A - B = \{1, 2\}$ and $B - A = \{6, 7\}$.
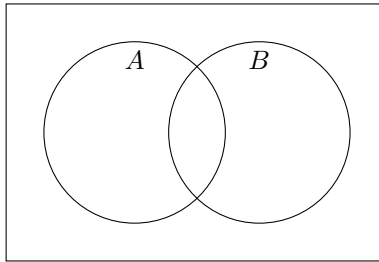
$\square$
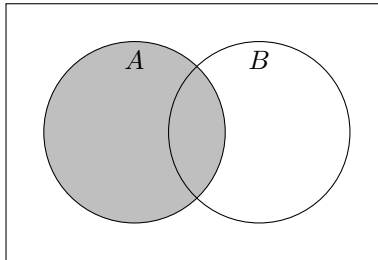
Note 3.4.1. Note that $A^c = A' = U - A$.

$\square$

4 Venn Diagrams

Useful for visualizing all of the above and for a good intuitive understanding of when things are true and not true. In a Venn diagram we start with a rectangle that indicates the universal set and typically another two sets A and B:
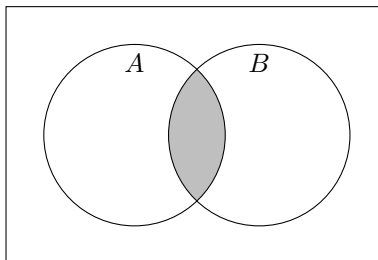

Then we can shade various regions:

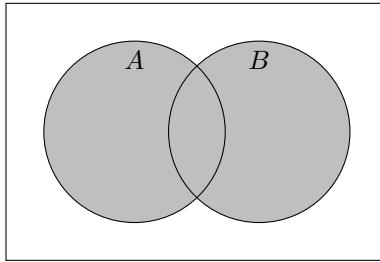**Example 4.1.** Here is A shaded:

$\square$

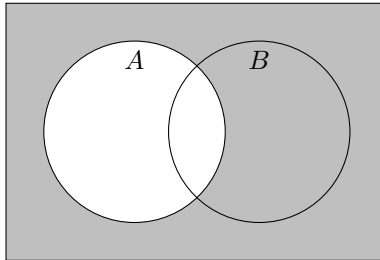**Example 4.2.** Here is $A \cap B$ shaded:

$\square$
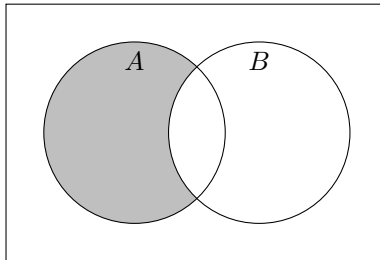
Example 4.3. Here is $A \cup B$ shaded:

□

Example 4.4. Here is A^c shaded:

□

Example 4.5. Here is $A - B$ shaded:

□

And lastly:

Example 4.6. Here is $A^c \cap B^c = (A \cup B)^c$ shaded:

□
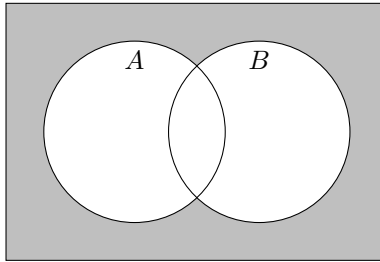
We can do the same with three sets.

Example 4.7. Here is $A - B - C$ shaded:

□
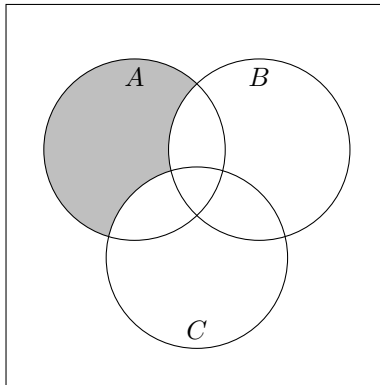
5   Partitions

Definition 5.0.1. A *partition* of a set A is a choice of dividing the elements of A into pairwise disjoint nonempty subsets whose union is A. This sounds complicated but it just means we're dividing up the elements of A in some way. Technically speaking a partition is a set of subsets.

□

Example 5.1. Suppose $A = \{1, 2, 3, 4, 5\}$. Here are some partitions of A:

- $\{\{1, 2\}, \{3, 4, 5\}\}$
- $\{\{1\}, \{2, 5\}, \{3, 4\}\}$
- $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
- $\{\{1, 2, 3, 4, 5\}\}$

The following are not partitions of A:

- $\{\{1,2\}, \{3,5\}\}$
- $\{\{1,2,3,4,5\}, \varnothing\}$
- $\{\{1,2,3\}, \{3,4,5\}\}$
- $\{1,2,3,4,5\}$

$\square$

6 Power Sets and Cartesian Products

These are useful ways of building new sets out of old sets.

6.1 Power Sets

Definition 6.1.1. For a set A, the *power set* of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

$\square$

Example 6.1. If $A = \{1,2\}$ then $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$

$\square$

Power sets can get pretty big.

Example 6.2. If $A = \{1,2,3,4\}$ then $\mathcal{P}(A)$ contains 16 elements. Do you see why?

$\square$

Theorem 6.1.1. If A has n elements then $\mathcal{P}(A)$ has 2^n elements.

—

Proof. When constructing a subset of A we can choose whether or not to put each element of A in that subset. We then make a binary choice n times for 2^n total possibilities. If this doesn't make complete sense right now it will make more sense when we get to combinatorics. $\mathcal{QED}$

$\square$

Power sets can be pretty confusing.

Example 6.3. We have:

- $\mathcal{P}(\varnothing) = \{\varnothing\}$

- $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

$\square$

Fun fact - power sets of the empty sets are one way that the nonnegative integers can be constructed from scratch!

$$\mathcal{P}(A) = \{S \,|\, S \subseteq A\}$$

6.2 Cartesian Products

Definition 6.2.1. If A and B are sets then we define *the cartesian product* $A \times B$ as:
$$A \times B = \{(a, b) \,|\, a \in A \wedge b \in B\}$$

$\square$

Example 6.4. If $A = \{1, 2\}$ and $B = \{\alpha, \beta\}$ then:

$$A \times B = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta)\}$$

$\square$

Example 6.5. Note that in mathematics we have the cartesian plane which is just the set of points (x, y). Formally this is $\mathbb{R} \times \mathbb{R}$ and is usually denoted $\mathbb{R}^2$.

$\square$

7 Cardinality and Countability

When studying sets it's useful to ask about how many things are in the set.

Definition 7.0.1. The cardinality of a set A, denoted $|A|$, loosely speaking, is the number of elements in a set.

$\square$

There is lots of formal structure to the study of cardinality but we will broadly break the cardinality of sets down first into two types and then one of those types will be broken down further.

The first division will be whether a set has finitely or infinitely many elements. If it is finite we can just give the number of elements.

Example 7.1. We have:

- $|\{1, 3, 5, 7\}| = 4$

- We have $|\mathbb{Z}|$ infinite

- We have $|\mathbb{R}|$ infinite.

$\square$

However the term "infinite" is vague and we will break it down further. We'll start by looking at two sets: $\mathbb{Z}^+$ and $\mathbb{R}$.

Observe that the set $\mathbb{Z}^+$ of positive integers may be listed one-by-one:

$$1, 2, 3, 4, 5, \ldots$$

It's not clear how we would do this with the set $\mathbb{R}$:

$$0.1, \pi, -45, 56.\overline{58}, \ldots?????$$

It turns out that these two sets have different sizes in a meaningful way. In order to formalize this we have the following definition:

Definition 7.0.2. We say that a set A is *countable* if we can match the element in 1-1 correspondance with $\mathbb{Z}^+$. Essentially this means we can choose a starting element and list all elements in a manner which does not miss any.

$\square$

Definition 7.0.3. We say that a set A is *uncountable* if it is not countable.

$\square$

Example 7.2. The set of positive even integers is countable. Here they are:

$$2, 4, 6, 8, \ldots$$

Formally the 1-1 matching is:

$\mathbb{Z}^+$		Positive Evens
1	$\leftrightarrow$	2
2	$\leftrightarrow$	4
3	$\leftrightarrow$	6
4	$\leftrightarrow$	8
$\vdots$	$\vdots$	$\vdots$

$\square$

Example 7.3. The set of all integers is countable. Here they are:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

Formally the 1-1 matching is:

$$\begin{array}{ccc} \mathbb{Z}^+ & & \mathbb{Z} \\ \hline 1 & \leftrightarrow & 0 \\ 2 & \leftrightarrow & 1 \\ 3 & \leftrightarrow & \text{-}1 \\ 4 & \leftrightarrow & 2 \\ 5 & \leftrightarrow & \text{-}2 \\ 6 & \leftrightarrow & 3 \\ \vdots & \vdots & \vdots \end{array}$$

$\square$

It is not clear right now whether we can or cannot do this with $\mathbb{R}$ but before we try, let's look at $\mathbb{Q}^+$, the set of positive rational numbers.

Theorem 7.0.1. The positive rationals are countable.

Proof. Observe that we can put all the positive rational numbers in an infinite table with the numerators n running across the top and the denominators d down the left edge:

n/d	1	2	3	4	...
1	1/1	2/1	3/1	4/1	...
2	1/2	2/2	3/2	4/2	...
3	1/3	2/3	3/3	4/3	...
4	1/4	2/4	3/4	4/4	...
5	1/5	2/5	3/5	4/5	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that there are repeats, for example 1/1 shows up as 2/2 and 1/2 shows up as 2/4 and so on.

Here's the fun part. We can use this table to generate a list of all of these numbers. There are various ways to do this but here is one:

Let's create a path through the table:

  (i) Start in the top left: $1/1$

 (ii) Go down one: $1/1, 1/2$

(iii) Go along the diagonal going up and right: $1/1, 1/2, 2/1$

 (iv) Go right one: $1/1, 1/2, 2/1, 3/1$

  (v) Go along the diagonal going down and left: $1/1, 1/2, 2/1, 3/1, 2/2, 1/3$

 (vi) Repeat from (ii).

This path hits everything in the table:

$$1/1, 1/2, 2/1, 3/1, 2/2, 1/3, 1/4, 2/3, 3/2, 4/1, 5/1, 4/2, 3/3, 2/4, 1/5, ...$$

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Now then, following this path, simply skip rationals that have previously been hit. This creates a list of all the positive rationals!

$$1/1, 1/2, 2/1, 3/1, 1/3, 1/4, 2/3, 3/2, 4/1, 5/1, 1/5, ...$$

$$\mathcal{QED}$$

$$\square$$

Okay, now let's go back to $\mathbb{R}$. We won't even look at all of them, let's just look at the reals in the interval $[0, 1]$.

Theorem 7.0.2. The set $[0, 1]$ is uncountable.

Proof. We proceed by induction. Assume we have a correspondance with $\mathbb{Z}^+$. Since each real number is a 0 followed by a string of decimal digits we can write our correspondance as follows. Let's assume we never run out of alphabet letters!

$$
\begin{array}{ccc}
\mathbb{Z}^+ & & \mathbb{Z} \\
1 & \leftrightarrow & 0.a_1 a_2 a_3 a_4 a_5 \ldots \\
2 & \leftrightarrow & 0.b_1 b_2 b_3 b_4 b_5 \ldots \\
3 & \leftrightarrow & 0.c_1 c_2 c_3 c_4 c_5 \ldots \\
4 & \leftrightarrow & 0.d_1 d_2 d_3 d_4 d_5 \ldots \\
5 & \leftrightarrow & 0.e_1 e_2 e_3 e_4 e_5 \ldots \\
\vdots & \vdots & \vdots
\end{array}
$$

Remember, we are assuming that every real number in $[0, 1]$ is in this list!

Consider the new real number built as follows:

$$0.\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \ldots$$

where we insist that:

- $\alpha_1 \neq a_1$, so this new number is not the first number in the list.
- $\alpha_2 \neq b_2$, so this new number is not the second number in the list.
- $\alpha_3 \neq b_3$, so this new number is not the third number in the list.
- $\alpha_4 \neq b_4$, so this new number is not the fourth number in the list.
- $\alpha_5 \neq b_5$, so this new number is not the fifth number in the list.
- ...etc...

Clearly this real number should be in the list by assumption, since the list is assumed to contain all of $[0, 1]$.

But this real number differs by the kth number at the kth digit, so it is not one of the numbers in the list.

This is a contradiction and we are done. $\hspace{2cm} \mathcal{QED}$

□

8 Set Identities

Much like the logical equivalences there are set identities which arise frequently and can be proved using the definitions. We have the following set identities. Some of them look similar to logical equivalences!

Commutative	$A \cup B = B \cup$ and $A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity	$A \cup \varnothing = A$ and $A \cap U = A$
Complement	$A \cup A^c = U$ and $A \cap A^c = \varnothing$
Double Complement	$(A^c)^c = A$
Idempotent	$A \cup A = A$ and $A \cap A = A$
Universal Bound	$A \cup U = U$ and $A \cap \varnothing = \varnothing$
DeMorgan's	$(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
Absorption	$A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
Complement of $\varnothing$ and U	$U^c = \varnothing$ and $\varnothing^c = U$
Set Difference	$A - B = A \cap B^c$

Later on we will discuss proving things with sets and we will be able to revisit these and prove them.

9 Set Theory Proofs (and Disproofs)

9.1 What Might We Prove (or Disprove)?

First of all we should note that our set-related proofs will fall into one of two categories:

(a) Proving: For all sets $A, B, \ldots$ some fact is true. Since this is a proof of a universal we will start with unknown sets $A, B, \ldots$ and go from there.

(b) Disproving: For all sets $A, B, \ldots$ some fact is true. This disproof of a universal becomes a proof of the negation which then involves finding explicit sets $A, B, \ldots$ for which the statement is false.

9.2 Proving or Disproving a Subset

Consider this example:

Example 9.1. Let's prove half of the set difference law $A - B = A \cap B^c$. Specifically let's prove that:

> For all sets A and B we have $A - B \subseteq A \cap B^c$.

$\square$

Before proceeding note that we want to prove $\subseteq$ so recall the definition. Here I've abstracted using S_1 and S_2 for the two sets:

$$S_1 \subseteq S_2 \text{ means } \forall x, x \in S_1 \to x \in S_2$$

We therefore start with some arbitrary $x \in S_1$ and show it is in S_2. Back to our example:

Example 9.2. We want to prove that:

For all sets A and B we have $A - B \subseteq A \cap B^c$.

Proof:

We start with some arbitrary $x \in A - B$.

We wish to prove that $x \in A \cap B^c$.

Since $x \in A - B$ we know that $x \in A$ and $x \notin B$ by the definition of $A - B$.

Since $x \notin B$ we know $x \in B^c$.

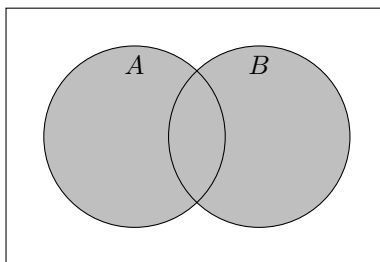Then since $x \in A$ and $x \in B^c$ we know $x \in a \cap B^c$.

$\square$

When disproving the subset property we look for a counterexample. A Venn diagram can help us but an explicit example is the final goal.
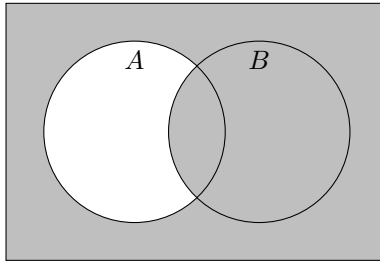
Example 9.3. Let's disprove:

For all sets A and B we have $A \cup B \subseteq A^c \cup B$.

Let's draw Venn Diagrams of the left and right sides. Here is $A \cup B$:

Here is $A^c \cup B$:
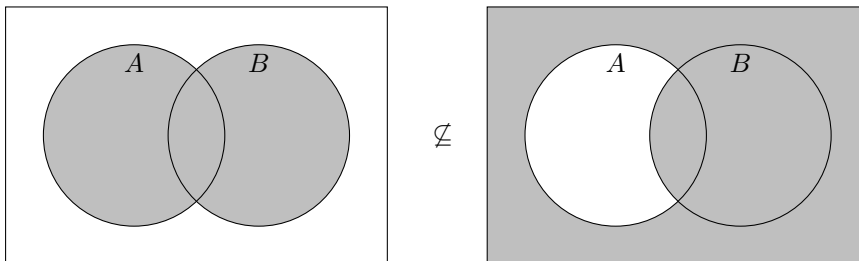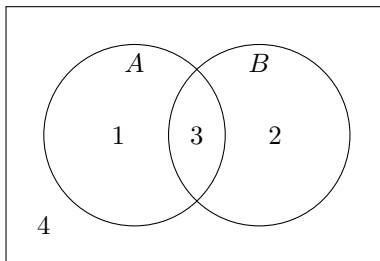
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Clearly we see:

  $\nsubseteq$

But we should give an explicit example. We can draw another Venn Diagram and put some numbers in the various areas:

So here is our actual (dis)proof:

(Dis)proof:

We assign:

$U = \{1, 2, 3, 4\}$

$A = \{1, 3\}$

$B = \{2, 3\}$

Then we have

$A \cup B = \{1, 2, 3\}$

$A^c \cup B = \{2, 3, 4\}$

And then $A \cup B \nsubseteq A^c \cup B$.

$\square$

9.3   Proving or Disproving Equality

When proving set equality we use the fact that:

$$S_1 = S_2 \longleftrightarrow (S_1 \subseteq S_2) \wedge (S_2 \subseteq S_1)$$

So we simply prove both of these. Specifically we prove:

$$\forall x, x \in S_1 \rightarrow x \in S_2$$

$$\text{and}$$

$$\forall x, x \in S_2 \rightarrow x \in S_1$$

Example 9.4. Let's prove the absorption law. Specifically let's prove that:

For all sets A and B, we have $A \cup (A \cap B) = A$.

Proof:

- First we prove that $A \cup (A \cap B) \subseteq A$.

  To do this, start with some $x \in A \cup (A \cap B)$.

  We wish to prove that $x \in A$.

  We know that either $x \in A$ or $x \in A \cap B$.

  If $x \in A$ then we are done.

  If $x \in A \cap B$ then $x \in A$ and we are done.

- Second we prove that $A \subseteq A \cup (A \cap B)$.

  To do this, start with some $x \in A$.

  We wish to prove that $x \in A \cup (A \cap B)$, however this is immediate since $x \in A$.

$\square$

Note 9.3.1. Note that we're using cases in the first bullet point. Remember cases:

$$
\begin{array}{l}
p \vee q \\
p \rightarrow r \\
q \rightarrow r \\
\therefore \quad r
\end{array}
$$

Do you see how? Cases emerge when our hypothesis breaks into two (or more) possibilities and we show that each of those leads to exactly the same conclusion.

Look at it this way. We proved that:

$$
\begin{array}{l}
(x \in A) \vee (x \in A \cap B) \\
(x \in A) \rightarrow (x \in A) \\
(x \in A \cap B) \rightarrow (x \in A) \\
\therefore \quad (x \in A)
\end{array}
$$

$\square$

Example 9.5. Let's prove one of the DeMorgan's Laws. Specifically let's prove that:

For all sets A and B we have $(A \cap B)^c = A^c \cup B^c$.

Proof:

- First we prove that $(A \cap B)^c \subseteq A^c \cup B^c$.

  To do this, start with some $x \in (A \cap B)^c$.

  We wish to prove that $A \in A^c \cup B^c$.

  Since $x \in (A \cap B)^c$ this means $x \notin A \cap B$ which then means either $x \notin A$ or $x \notin B$ (or both).

  If $x \notin A$ then $x \in A^c$ and then $x \in A^c \cup B^c$.

  if $x \notin B$ then $x \in B^c$ and then $x \in A^c \cup B^c$.

- Second we prove that $A^c \cup B^c \subseteq (A \cap B)^c$. To do this, start with some $x \in A^c \cup B^c$.

  We wish to prove that $x \in (A \cap B)^c$.

  Since $x \in A^c \cup B^c$ we know either $x \in A^c$ or $x \in B^c$ (or both).

  If $x \in A^c$ then $x \notin A$ and then $x \notin A \cap B$ and then $A \in (A \cap B)^c$.

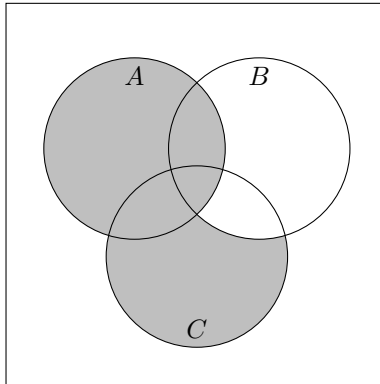 If $x \in B^c$ then $x \notin B$ and then $x \notin A \cap B$ and then $A \in (A \cap B)^c$.

$\square$

When disproving set equality we look for a counterexample. That counterexample only needs to violate one of the two subset statements! A Venn diagram can help us but an explicit example is the final goal.
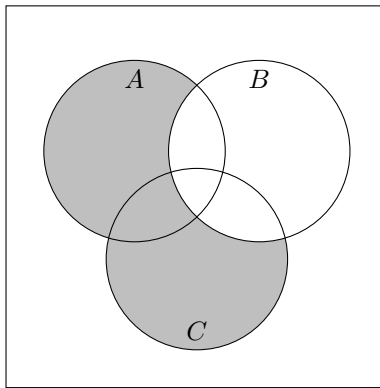
Example 9.6. Let's disprove:

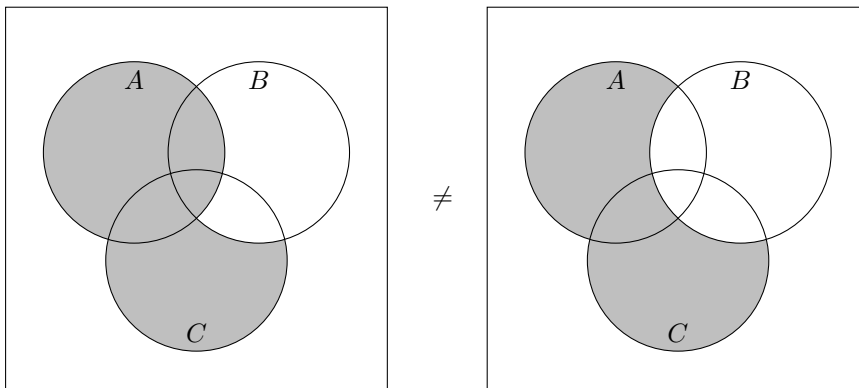For all sets A, B, and C we have $A \cup (C - B) = (A \cup C) - B$.

Let's draw Venn Diagrams of the left and right sides. Here is $A \cup (C - B)$:
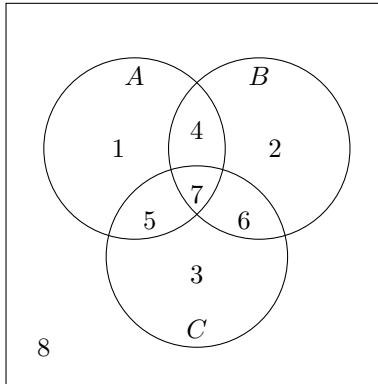
Here is $(A \cup C) - B$:

Clearly we see:

Specifically $A \cup (C - B) \not\subseteq (A \cup C) - B$.

But we should give an explicit example. We can draw another Venn Diagram and put some numbers in the various areas:

So here is our actual (dis)proof:

(Dis)proof:

We assign:

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 4, 5, 7\}$

$B = \{2, 4, 6, 7\}$

$C = \{3, 5, 6, 7\}$

Then we have

$A \cup (C - B) = \{1, 3, 4, 5, 7\}$

$(A \cup C) - B = \{1, 3, 5\}$

And then $A \cup (C - B) \neq (A \cup C) - B$.

$\square$

9.4 Proving $A = \varnothing$

When proving $A = \varnothing$ we proceed differently because the logic above doesn't apply.

Instead, we prove this by contradiction - we assume that there is something in A and find a contradiction.

Example 9.7. Let's prove that:

$$\text{For any set } A \text{ that } A \cap \varnothing = \varnothing.$$

Suppose not, so that there is some $x \in A \cap \varnothing$. Then we know $x \in A$ and $x \in \varnothing$. But the second of these is impossible and we have our contradiction.

$\blacksquare$

9.5   Proving Conditionals Related to Sets

When we are proving conditionals related to sets we have to start pulling several things together.

Example 9.8. Let's prove that:

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \varnothing$.

Proof:

This is a for all statement so we start with unknown sets A, B, and C and we assume the hypotheses, that $A \subseteq B$ and $B \subseteq C^c$.

We then claim that $A \cap C = \varnothing$.

To prove $A \cap C = \varnothing$ we proceed by contradiction and assume $x \in A \cap C$.

Since $x \in A \cap C$ we know that $x \in A$ and $x \in C$.

Since $x \in A$ and $A \subseteq B$ and $B \subseteq C^c$ we know $x \in C^c$.

However then $x \in C$ and $x \in C^c$ and this is a contradiction.

$\square$