CMSC 250: Weak and Strong Mathematical Induction

Justin Wyss-Gallifent

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1 Weak Induction Introduction

Here are two hypothetical situations that can help communicate the idea of induction.

1.1 A Domino Argument

Suppose there are infinitely many dominoes labeled 1,2,3,... standing up in such a way that when you push over domino k it then pushes over domino $k+1$. Of course nothing is happening yet. Suppose then you push over domino 1. What will happen? Well, domino 1 falls, causing domino 2 to fall, causing domino 3 to fall, and so on. For every n, domino n will fall.

In this hypothetical there are two things that are true. Let's say $F(n)$ is true iff domino n falls.

(a) The first domino falls. That is, $F(n)$ is true.

(b) For every domino k, if domino k falls then domino $k+1$ falls. that is, $\forall k \geq 1, F(k) \rightarrow f(k+1)$.

You can conclude that $\forall n \geq 1, F(n)$.

1.2 A Weather Argument

Suppose I told you two things:

(a) It will be sunny on January 1 2022.

(b) For any day from January 1 2022 onwards, if it is sunny on that day, it will be sunny on the next day.

What could you conclude?

You could conclude that it will be sunny every day from January 1 2022 onwards!

The reason for this is that:

- First (a) tells you it will be sunny on January 1 2022.
- Then (b) tells you that because it it sunny on January 1 2022, it will also be sunny on January 2 2022.
- Then (b) tells you that because it it sunny on January 2 2022, it will also be sunny on January 3 2022.
- Then (b) tells you that because it it sunny on January 3 2022, it will also be sunny on January 4 2022.
- And so on.

The idea behind weak mathematical induction is just this. Strong mathematical induction is only slightly different.

2 Weak Mathematical Induction

2.1 Introduction

Weak mathematical induction is also known as the First Principle of Mathematical Induction and works as follows:

2.2 How it Works

Suppose some statement $P(n)$ is defined for all $n \geq n_0$ where n_0 is a nonnegative integer. Suppose that we want to prove that $P(n)$ is actually true for all $n \geq n_0$. We do this by proving two things:

(a) The Base Case: We prove that $P(n_0)$ is true.

(b) The Inductive Step: We prove that for any $k \geq n_0$, if $P(k)$ is true (this is called the inductive hypothesis) then $P(k+1)$ is also true.

More formally we have:

Theorem 2.2.1. Suppose $P(n)$ is a statement for all $n \geq n_0$ and suppose that:

(a) $P(n_0)$

(b) $\forall k \geq n_0, P(k) \rightarrow P(k+1)$

Then $\forall n \geq n_0, P(n)$.

2.3 Examples

Example 2.1. Let's prove that $2^n - n^2 \geq 0$ for all $n \geq 4$.

Here $P(n)$ is the statement $2^n - n^2 \geq 0$ and the starting value is $n_0 = 4$.

(a) The Base Case:

We claim that $2^4 - 4^2 \geq 0$. Well $2^4 - 4^2 = 16 - 16 = 0 \geq 0$ so it is true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 4,\ \text{if } 2^k - k^2 \geq 0 \text{ then } 2^{k-1} - (k+1) \geq 0$$

Suppose that $k \geq 4$ and $2^k - k^2 \geq 0$ (the inductive hypothesis).

Observe that:

$$
\begin{aligned}
2^{k+1} - (k+1)^2 &= 2 \cdot 2^k - (k+1)^2 \\
&\geq 2k^2 - (k+1)^2 \\
&= 2k^2 - k^2 - 2k - 1 \\
&= k^2 - 2k - 1 \\
&= k(k-2) - 1 \geq 4(4-2) - 1 = 7 \geq 0
\end{aligned}
$$

Then we are done.

Example 2.2. Let's prove that:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for all $n \geq 0$.

(a) The Base Case:

We claim that:

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

Since the left hand side is 1 and the right hand side is 1 this is true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 1, \text{ if } \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \text{ then } \sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$$

Suppose that $k \geq 1$ and $\sum_{i=1}^{k} = \frac{k(k+1)}{2}$ (the inductive hypothesis).

Observe that:

$$\begin{aligned}
\sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2} &= \left[\sum_{i=1}^{k} i\right] + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2}
\end{aligned}$$

Then we are done.

Example 2.3. Suppose that $n \geq 1$. Take a $2^n \times 2^n$ chessboard with a corner removed. Let's prove the board can be covered with three-square L-shaped pieces.

(a) The Base Case:

This obviously true for a $2^1 \times 2^1$ chessboard with a corner removed because it takes exactly one L-shaped piece to cover it.

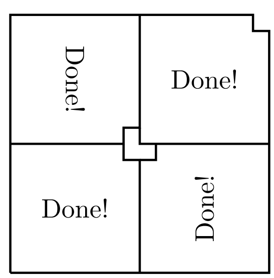
(b) The Inductive Step:

We will prove that:

For all $k \geq 1$, if we can cover a $2^k \times 2^k$ chessboard with a corner removed then we can cover a $2^{k+1} \times 2^{k+1}$ chessboard with a corner removed.

Suppose we have (somehow) managed to cover a $2^k \times 2^k$ chessboard with a corner removed (the inductive hypothesis):

If we're now given a $2^{k+1} \times 2^{k+1}$ chessboard with a corner removed we can cover it using three of the above arrangments (whatever they are) and one extra L-shaped piece. Notice that this new chessboard is twice as long and twice as wide as the previous one.



Then we are done.

Example 2.4. Let's prove that:

$$\sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$$

for all $n \geq 1$.

(a) The Base Case:

We claim that:

$$\sum_{j=1}^{1} j(j+1) = \frac{1(1+1)(1+2)}{3}$$

Since the left hand side is 2 and the right hand side is 2 it is true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 1, \text{ if } \sum_{j=1}^{k} j(j+1) = \frac{k(k+1)(k+2)}{3}$$

$$\text{then } \sum_{j=1}^{k+1} j(j+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

Suppose that $k \geq 1$ and $\sum_{j=1}^{k} j(j+1) = \frac{k(k+1)(k+2)}{3}$ (the inductive hypothesis).

Observe that:

$$\begin{aligned}
\sum_{j=1}^{k+1} j(j+1) &= \left[\sum_{j=1}^{k} j(j+1)\right] + (k+1)((k+1)+1) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}$$

Then we are done.

Example 2.5. Define a sequence recursively by:

$$a_n = \begin{cases} a_0 = 3 \\ a_n = 5a_{n-1} + 8 \quad \text{For } n \geq 1 \end{cases}$$

Let's prove that $a_n \equiv 3 \mod 4$ for all $n \geq 0$.

(a) The Base Case:

We claim that $a_0 \equiv 3 \mod 4$ but since $a_0 = 3$ this is clearly true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 0, \text{ if } a_k \equiv 3 \mod 4 \text{ then } a_{k+1} \equiv 3 \mod 4$$

Suppose that $k \geq 0$ and $a_k \equiv 3 \mod 4$ (the inductive hypothesis).

Observe that:

$$\begin{aligned} a_{k+1} &= 5a_k + 8 \\ &\equiv 5(3) + 8 \mod 4 \\ &\equiv 23 \mod 4 \\ &\equiv 3 \mod 4 \end{aligned}$$

Then we are done.

Example 2.6. Let's prove that $3 \mid n^3 - n$ for all $n \geq 1$.

(a) The Base Case:

We claim that $3 \mid 1^3 - 1$ but since $1^3 - 1 = 0$ and $3 \mid 0$ this is clearly true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 0, \text{ if } 3 \mid k^3 - k \text{ then } 3 \mid (k+1)^3 - (k+1)$$

Suppose that $3 \mid k^3 - k$ (the inductive hypothesis). This means that $k^3 - k = 3\alpha$ for some $\alpha \in \mathbb{Z}$.

Observe that:

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\
&= k^3 - k + 3k^2 + 3k \\
&= 3\alpha + 3(k^2 + k) \\
&= 3(\alpha + k^2 + k)
\end{aligned}$$

And so $3 \mid (k+1)^3 - (k+1)$.

Then we are done.

Example 2.7. Let's prove that:

$$1(1!) + 2(2!) + ... + n(n!) = (n + 1)! - 1$$

For all $n \geq 0$.

(a) The Base Case:

We claim that $1(1!) = (1 + 1)! - 1$. Since the left hand side is 1 and the right hand side is 1 this is true.

(b) The Inductive Step:

We will prove that:

$$\forall k \geq 0, \text{ if } 1(1!) + 2(2!) + ... + k(k!) = (k + 1)! - 1$$

$$\text{then } 1(1!) + 2(2!) + ... + (k + 1)(k + 1)! = ((k + 1) + 1)! - 1$$

Suppose that $1(1!) + 2(2!) + ... + k(k!) = (k + 1)! - 1$ (the inductive hypothesis).

Then observe that:

$$\begin{aligned} 1(1!) + 2(2!) + ... + (k + 1)(k + 1)! &= 1(1!) + 2(2!) + ... + k(k!) + (k + 1)(k + 1)! \\ &= (k + 1)! - 1 + (k + 1)((k + 1)!) \\ &= 1(k + 1)! + (k + 1)(k + 1)! - 1 \\ &= (1 + k + 1)(k + 1)! - 1 \\ &= ((k + 2)(k + 1)! - 1 \\ &= (k + 2)! - 1 \end{aligned}$$

Then we are done.

3 Strong Mathematical Induction

3.1 Introduction

Let's begin with an intutive example. This is not a formal proof by strong induction (we haven't even talked about what strong induction is!) but it hits some of the major ideas intuitively.

Example 3.1. Suppose that all we have are 3¢ and 10¢ stamps. Prove that we can make any postage of 18¢ or more.

The first thing to note is that if we tried to use weak induction the inductive step won't help. This is because knowing that we can make exactly k¢ doesn't give us any information about how to make $(k+1)$¢ .

Suppose we started doing it one by one:

$$\begin{aligned}
18 &= 3+3+3+3+3+3 \\
19 &= 10+3+3+3 \\
20 &= 10+10
\end{aligned}$$

At this point we might notice something We might notice that to do 21¢ we can simply do 18¢ and add a 3¢ stamp. Then to do 22¢ we simply do 19¢ and add a 3¢ stamp. Since this idea continues see that we can in fact do any postage 18¢ or more.

Before discussing strong mathematical induction formally we will state that the three cases we did first are the three base cases and that the thing we notice is the inductive step.

Observe that all three base cases were necessary because we can't try to do 20¢ by doing 17¢ and adding a 3¢ stamp because we haven't done 17¢ , and in fact 17¢ can't be done!

3.2 How it Works

The general idea behind strong mathematical induction is this.

(a) We prove some number of base cases n_0, ... ,$n =$???.

(b) We assume that the statement is true for $i = n_0, n_0+1, ..., k$ and we prove that the statement is true for $k+1$.

The major point of confusion arises because it may not be clear how many base cases we need to prove. In light of this it is usually easier to prove the inductive step first and then check the inductive step to see how far back it references. That refence must be greater than or equal to n_0 and that will tell us how far our base cases need to go.

Confused? Let's return to the stamp problem again and approach it from this angle.

3.3 Examples

Example 3.2. Suppose that all we have are 3¢ and 10¢ stamps. Prove that we can make any postage of 18¢ or more.

(a) The Inductive Step:

Assume we can make all amounts $18, 19, 20, ..., k$ (the induction hypothesis). How can we make $k + 1$? Well, easy, we simply make $(k - 2)$¢ and add in another 3¢ stamp.

(b) The Base Case(s):

In order to do what we did, we must have $k - 2 \geq 18$ which means $k \geq 20$.

This means that the cases $18, 19, 20$ must be done separately and induction gets us from 20 to 21, from 21 to 22, and so on.

So we have three base cases:

$$
\begin{aligned}
18 &= 3 + 3 + 3 + 3 + 3 + 3 \\
19 &= 10 + 3 + 3 + 3 \\
20 &= 10 + 10
\end{aligned}
$$

Then we are done.

Note 3.3.1. I have seen very few resources which suggest doing the inductive step first and using it to analyze how many base cases are needed. Most resources just, somehow, produce the number of base cases out of thin air, which is confusing.

Example 3.3. Define a sequence recursively by:

$$a_n = \begin{cases} a_1 = 1 \\ a_2 = 4 \\ a_n = 2a_{n-1} - a_{n-2} & \text{For } n \geq 3 \end{cases}$$

Let's prove that $a_n = 3n - 2$ for all $n \geq 1$.

(a) The Inductive Step:

We assume that $a_i = 3i - 2$ for $i = 1, 2, ..., k$ and we prove that $a_{k+1} = 3(k+1) - 2$ (the induction hypothesis).

Observe that:

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 2(3k-2) - (3(k-1) - 2) \\ &= 6k - 4 - 3k + 3 + 2 \\ &= 3k + 1 \\ &= 3(k+1) - 2 \end{aligned}$$

(b) The Base Case(s):

When proving the statement for $k + 1$ we had to use the statement for $k - 1$. Because our claim starts with $n = 1$ we have to ensure that $k - 1 \geq 1$, so $k \geq 2$.

This means that there are two base cases, a_1 and a_2. We check those:

$a_1 = 1 = 3(1) - 2$ is true and $a_2 = 4 = 3(2) - 2$ is true.

Then we are done.

Example 3.4. Let's prove that every integer greater than 2 can be written as a product of (perhaps just one) primes.

(a) The Inductive Step:

Suppose every integer $2, 3, ..., k$ can be written as the product of primes (the induction hypothesis). We claim that $k + 1$ can be, also.

Either $k + 1$ is prime or it isn't. If it's prime, then it can be written as a product of itself.

If it isn't prime then $k + 1 = ab$ with $2 \leq a \leq k$ and $2 \leq b \leq k$. By the induction hypothesis we know a and b are each products of primes, and therefore $k + 1$ is, too.

(b) The Base Case(s):

The base cases are a bit sneaky here. In order to write $k + 1 = ab$ with $2 \leq a \leq k$ and $2 \leq b \leq k$ we must have $k + 1 \geq (2)(2) = 4$. Thus we must have $k \geq 3$ and so our base cases are 2 and 3.

Each of these can certainly be written as the product of one prime.

Then we are done.