CMSC 351: Big Notation

Justin Wyss-Gallifent

February 11, 2021

1 The Bigs ................................................................. 2
  1.1 Big-O Notation .................................................. 2
  1.2 Big-Omega and Big-Theta Notations ......................... 3
  1.3 All Together ..................................................... 4

2 Additional Facts ....................................................... 6
  2.1 Use of n vs x ..................................................... 6
  2.2 Cautious Comparisons ......................................... 6
  2.3 Nice Functions .................................................. 7
  2.4 Alternative Formulations ..................................... 8

3 Thoughts, Problems, Ideas .......................................... 9
1 The Bigs

1.1 Big-O Notation

Recall the definition:

**Definition 1.1.1.** We say that:

\[ f(x) = O(g(x)) \text{ if } \exists x_0, C > 0 \text{ such that } \forall x \geq x_0, f(x) \leq Cg(x) \]

We think of this as stating that *eventually* \( f(x) \) is smaller than some constant multiple of \( g(x) \).

**Example 1.1.** For example, here \( f(x) = O(x^2) \) with \( C = 2 \) and \( x_0 \) as shown:

![Graph showing function f(x) and Cg(x) with x0 as the threshold](image)

**Note 1.1.1.** There’s frequently (but not always) a trade-off in that if \( C \) is large then \( x_0 \) might be smaller, or vice-versa. In light of this note that “eventually” could mean for a very large \( x_0 \).

**Example 1.2.** It’s true that \( 42000x \log x = O(x^2) \) with \( C = 10 \) because eventually \( 42000x \log x \leq 10x^2 \). However “eventually” in this case means \( x_0 \approx 67367 \). In other words this is the smallest \( x_0 \) such that if \( x \geq x_0 \) then \( 42000x \log x \leq 10x^2 \).
1.2 Big-Omega and Big-Theta Notations

We can extend upon this with:

**Definition 1.2.1.** We have:

\[ f(x) = \Omega(g(x)) \text{ if } \exists x_0, B > 0 \text{ such that } \forall x \geq x_0, f(x) \geq Bg(x) \]

**Example 1.3.** For example, here \( f(x) = \Omega(x^2) \) with \( B = \frac{1}{2} \) and \( x_0 \) as shown:

and with:

**Definition 1.2.2.** We have:

\[ f(x) = \Theta(g(x)) \text{ if } \exists x_0, B > 0, C > 0 \text{ such that } \forall x \geq x_0, Bg(x) \leq f(x) \leq Cg(x) \]

**Example 1.4.** For example, here \( f(x) = \Theta(x^2) \) with \( B = \frac{1}{2} \) and \( C = 2 \) and \( x_0 \) as shown:
1.3 All Together

The basic idea is that $O$ provides an upper bound for $f(x)$, $\Omega$ provides a lower bound and $\Theta$ provides a tight bound. Therefore $f(x) = \Theta(g(x))$ if and only if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

Moreover observe that $\Theta \Rightarrow O$ and $\Theta \Rightarrow \Omega$ but the converses are false.

Example 1.5. We show: $3x \lg x + 17 = O(x^2)$

Consider the expression:

$$3x \lg x + 17$$

Note two things:

- If $x > 0$ then $\lg x < x$.
- If $x \geq \sqrt{17} = 4.1231...$ then $x^2 \geq 17$.

Thus if $x \geq 5$ both of these are true and we have:

$$3x \lg x + 17 \leq 3x(x) + x^2 = 4x^2$$

Thus $x_0 = 5$ and $C = 4$ works.

Note: It’s not necessary to pick an integer value of $x_0$ here. I just did it because it’s pretty. Using $x_0 = \sqrt{17}$ would have been fine too.

Example 1.6. We show: $\frac{100}{x^2} + x^2 \lg x = O(x^3)$

Consider the expression:

$$\frac{100}{x^2} + x^2 \lg x$$

Note two things:

- If $x > 0$ then $\lg x < x$.
- If $x \geq 10$ then $x^2 \geq 100$ and then $\frac{100}{x^2} \leq 1 < x < x^3$.

Thus if $x \geq 10$ both of these are true and we have:

$$\frac{100}{x^2} + x^2 \lg x = O(x^3) \leq x^3 + x^3 = 2x^3$$

Thus $x_0 = 10$ and $C = 2$ works.

Example 1.7. We show: $0.001x \lg x + 0.0001x - 42 = \Omega(x)$

Consider the expression:

$$0.001x \lg x - 42$$

Note that if $x \geq 2$ then $\lg x \geq 1$ and then:

$$0.001x \lg x - 42 \geq 0.001x - 42$$
This is a line with slope 0.001 and any line with smaller slope will eventually be below it. For example the line 0.0001$x$ is below it when:

\[
0.001x - 42 \geq 0.0001x \\
0.0009x \geq 42 \\
x \geq \frac{42}{0.0009} = 46666.66...
\]

Thus if we have \(x \geq 46666.66\) then:

\[
0.001x \lg x - 42 \geq 0.001x - 42 \geq 0.0001x
\]

Thus \(x_0 = 46667\) and \(B = 0.0001\) works.

**Example 1.8.** We show: \(10x \lg x + x^2 = \Theta(x^2)\)

Consider the expression:

\[
10x \lg x + x^2
\]

Observe that for all \(x \geq 1\) we have \(\lg x > 0\) and hence:

\[
10x \lg x + x^2 \geq x^2
\]

And we have:

\[
10x \lg x + x^2 \leq 10x(x) + x^2 = 11x^2
\]

Thus \(x_0 = 1\), \(B = 1\) and \(C = 11\) works.

For simple polynomials there’s very little work to show \(\Theta\).

**Example 1.9.** Observe that \(3x^2 = \Theta(x^2)\) because \(x_0 = 0\) and \(B = C = 3\) works.

**Example 1.10.** Consider \(f(x) = 2x^2 - x\). Note that \(2x^2 - x \leq 2x^2\) and \(2x^2 - x \geq 2x^2 - x^2 = 1x^2\) for \(x \geq 1\) so that \(x_0 = 1, B = 1, C = 2\) works for \(2x^2 - x = \Theta(x^2)\).

**Example 1.11.** Consider \(f(x) = 0.001x^2(1 + \cos(x\pi))\).

The graph of this function is:
The local maxima occur at \( x = 0, 2, 4, 6, 8, \ldots \) and the local minima occur at \( x = 1, 3, 5, 7, 9, \ldots \).

Note that \( 0.001x^2(1 + \cos(x\pi)) \leq 0.001x^2(1 + 1) = 0.002x^2 \) for \( x \geq 0 \) so that \( f(x) = O(x^2) \). However in addition note that when \( x \in \mathbb{Z} \) is odd that \( 0.001x^2(1 + \cos(x\pi)) = 0.001x^2(1 - 1) = 0 \) so that there is no \( B > 0 \) such that for large enough \( x \) we have \( f(x) \geq Bx^2 \). Consequently \( f(x) \neq \Omega(x^2) \) and thus \( f(x) \neq \Theta(x^2) \).

You might ask if there is any \( g(x) \) such that \( f(x) = \Theta(g(x)) \) and the short answer is - yes, of course, because \( f(x) = \Theta(f(x)) \) but this is generally unsatisfactory. We are looking for useful \( g(x) \) which help us understand \( f(x) \). Saying essentially that \( f(x) \) grows at the same rate as itself doesn’t help much!

## 2 Additional Facts

### 2.1 Use of \( n \) vs \( x \)

These statements about function of \( x \) are often phrased using the variable \( n \) instead. Typically this is done when \( n \) can only take on positive integers.

In this case it can still be helpful to draw the functions as if \( n \) could be any real number, otherwise we’re left drawing a bunch of dots. In some cases though, like \( f(n) = n! \), it’s not entirely clear how we would sketch this for \( n \notin \mathbb{Z} \).

Otherwise the calculations are basically identical, noting that the cutoff value \( n_0 \) must be a positive integer.

### 2.2 Cautious Comparisons

This notation brings a certain ordering to functions. Observe for example that \( 1000000 + n \lg n = O(n^2) \) because eventually \( 1000000 + n \lg n \leq Cn^2 \) for some \( C > 0 \). Thus we intuitively think of \( n^2 \) as “larger than” \( 1000000 + n \lg n \). However we have to make sure we understand that we really mean that a constant multiple of \( n^2 \) is eventually larger than \( 1000000 + n \lg n \).
Example 2.1. For example, eventually $1000000 + n \lg n \leq 17n^2$ but eventually here means for $n \geq n_0 = 243$.

We should also note that it’s common to believe that if one function $g(x)$ has a larger derivative than another function $f(x)$ that eventually $f(x) \leq g(x)$. This is false.

2.3 Nice Functions

The functions we tend to compare against are “nice” in the sense that they capture a certain sense of growth. For this reason they tend to be increasing functions. Some examples are $\lg n$, $n \lg n$, $n^2$, and so on. We’ll see more later.
2.4 Alternative Formulations

There are a few alternative ways of proving $O$, $\Omega$ and $\Theta$. Here is one. Note that the following are unidirectional implications!

**Theorem 2.4.1.** Provided $\lim_{n \to \infty} f(n)$ and $\lim_{n \to \infty} g(n)$ exist (they may be $\infty$) then we have the following:

(a) If $\lim_{n \to \infty} \frac{f(n)}{g(n)} \neq 0, \infty$ then $f(n) = \Theta(g(n))$.

Note: Here we also have $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ as well.

(b) If $\lim_{n \to \infty} \frac{f(n)}{g(n)} \neq \infty$ then $f(n) = O(g(n))$.

(c) If $\lim_{n \to \infty} \frac{f(n)}{g(n)} \neq 0$ then $f(n) = \Omega(g(n))$.

**Proof.** Omitted. \( \square \)

**Example 2.2.** Observe that:

$$\lim_{n \to \infty} \frac{n \ln n}{n^2} = \lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{1/n}{1} = 0$$

Thus $n \ln n = O(n^2)$.

The following example is far easier to prove using this theorem than from the definition of $O$:

**Example 2.3.** We have $50n^{100} = O(3^n)$.

Observe that 100 applications of L’hôpital’s Rule yields:

$$\lim_{n \to \infty} \frac{50n^{100}}{3^n} = \lim_{n \to \infty} \frac{(100)(99)...(1)(50)}{(\ln 3)^{100}3^n} = 0$$

The result follows.
3 Thoughts, Problems, Ideas

1. It’s tempting to think that if $f(x)$ and $g(x)$ are both positive functions defined on $[0, \infty)$ with positive derivatives and if $f'(x) > g'(x)$ for all $x$ then eventually $f(x)$ will be above $g(x)$. Show that this isn’t true. Give explicit functions and sketches of those functions.

2. Find the value $x_0$ (approximately) which justifies $1234 + 5678 \log x = \mathcal{O}(x^2)$ with $C = 42$. Use any technology you like but explain your process.

3. Find the value $x_0$ (approximately) which justifies $4758x + 789x^2 \log x = \mathcal{O}(x^3)$ with $C = 17$. Use any technology you like but explain your process.

4. Find the value $x_0$ (approximately) which justifies $0.00357x^{2.01} \log x = \Omega(x^2)$ with $C = 100$. Use any technology you like but explain your process.

5. Show from the definition that $5x^2 + 10x \log x + \log x = \mathcal{O}(x^2)$.

6. Show from the definition that:
   \[
   \sum_{i=0}^{n-1} \left[ 2 + \sum_{j=i}^{n-1} 3 \right] = \mathcal{O}(n^2)
   \]

7. Show from the definition that $(475632)2^n = \mathcal{O}(5^n)$.

8. Show from the definition that $x + x \log x = \Theta(x \log x)$.

9. Show from the definition that:
   \[
   \sum_{i=0}^{n-1} \left[ 1 + i + \frac{1}{i+1} \right] = \Theta(n^2)
   \]

10. Show from the definition that $x^3 + 5x + \ln x + 100 = \Omega(x^2)$.

11. Show from the definition that:
    \[
    \sum_{i=0}^{n} [i^2 + 3i] = \Omega(n^3)
    \]

12. Show from the definition that $5^n \neq \mathcal{O}(2^n)$.

13. Show that $5000 + 6000n^{1500} = \mathcal{O}(3^n)$.

14. Show that $5^n = \Omega(n^{1000})$

15. Show that $(0.001)5^n = \Omega(857n^{999})$

16. Show from the definition that $\log_2 n = \Theta(\log_5 n)$ and $\log_5 n = \Theta(\log_2 n)$. 
17. Generalize the above problem. In other words prove that \( \Theta(\log_b x) = \Theta(\log_c x) \) for any two bases \( b, c > 1 \).

18. In the previous question why do we need \( b, c > 1 \)?

19. Give an example of two functions \( f(x) \) and \( g(x) \) which are not constant multiples of one another and which satisfy \( f(x) = O(g(x)) \) and \( g(x) = O(f(x)) \). Justify from the definitions.

20. Give an example of two functions \( f(x) \) and \( g(x) \) which are not constant multiples of one another and which satisfy \( f(x) = \Omega(g(x)) \) and \( g(x) = \Omega(f(x)) \). Justify from the definitions.

21. If \( f(n) = O(g(n)) \) with \( C_0 \) and \( n_0 \) and \( g(n) = O(h(n)) \) with \( C_1 \) and \( n_1 \) which constants would prove that \( f(n) = O(h(n)) \)?

22. The functions \( f(x) = \log_b x \) for \( b > 1 \) and \( g(x) = x^c \) for \( 0 < c < 1 \) have similar shapes for increasing \( x \). however \( f(x) = O(g(x)) \) always. Prove this.
   Note: This underlies the important fact that roots always grow faster than logarithms.

23. Consider the following three functions:

\[ f(x), g(x), h(x) \]

(a) Write down as many possibilities as you can which satisfy \( \square = O(\hat{\diamond}) \) where \( \square, \hat{\diamond} \in \{ f(x), g(x), h(x) \} \).

(b) Write down as many possibilities as you can which satisfy \( \square = \Omega(\hat{\diamond}) \) where \( \square, \hat{\diamond} \in \{ f(x), g(x), h(x) \} \).

(c) Write down as many possibilities as you can which satisfy \( \square = \Theta(\hat{\diamond}) \) where \( \square, \hat{\diamond} \in \{ f(x), g(x), h(x) \} \).