1 Introduction

B-trees are generalizations of 2-3 trees in which the number of keys and children is permitted to vary.

2 Definition

Definition 2.0.1. For an integer $m \geq 3$ a B-tree of order $n$ is a multiway search tree with the following properties:

- The root is either a leaf (a single node tree) or has between 2 and $m$ children.
- Each node except the root has between $\lceil m/2 \rceil$ and $m$ children, which may be null.
- A node with $j$ children contains $j - 1$ keys.
- All leaf nodes are at the same level.

By a multiway search tree we mean a generalization of BSTs and 2-3 trees. If a node has keys $a_1 < a_2 < \ldots < a_k$ then the children are roots of subtrees $A_1, \ldots, A_{k+1}$ with keys satisfying $A_1 < a_1 < A_2 < a_2 < \ldots < a_k < A_{k+1}$.

Note 2.0.1. Observe that by definition a B-tree is perfect, meaning it is full on every level.

Example 2.1. Here is an example of a B-tree of order $m = 5$. The root must have between 2 and $m = 5$ children (hence 1 and 4 keys) while the other nodes must have between $\lceil m/2 \rceil = \lceil 5/2 \rceil = 3$ and 5 children (hence 2 and 4 keys). The leaf nodes are shown as vertical just for spacing reasons.
3 Height

As with many of our other trees the height is logarithmic as a function of the number of nodes and keys. Here is the proof for keys, which is more relevant since typically we are interested in key counts:

**Theorem 3.0.1.** Suppose B-tree of order \( m \) has \( k \) keys and height \( h \). Then:

(a) The sparsest possible such tree has:

\[
k = 2 \lceil m/2 \rceil^h - 1
\]

(b) Consequently any such tree has:

\[
k \geq 2 \lceil m/2 \rceil^h - 1
\]

(c) And it then follows that:

\[
h \leq \log_{\lceil m/2 \rceil} \left( \frac{k + 1}{2} \right)
\]

(d) Thus:

\[
h = O(\log k)
\]

**Proof.** To obtain the maximum height we need the fewest keys allowed per node. The following table illustrates the minimum number of nodes and keys for each level 0 (the root) through \( h \) (the leaves).

<table>
<thead>
<tr>
<th>Level</th>
<th>Min Nodes</th>
<th>Min Keys</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1 ( \lceil m/2 \rceil - 1 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2 ( \lceil m/2 \rceil - 1 )</td>
</tr>
<tr>
<td>2</td>
<td>2 ( \lceil m/2 \rceil )</td>
<td>2 ( \lceil m/2 \rceil ) ( \lceil m/2 \rceil - 1 )</td>
</tr>
<tr>
<td>3</td>
<td>2 ( \lceil m/2 \rceil^2 )</td>
<td>2 ( \lceil m/2 \rceil^2 ) ( \lceil m/2 \rceil - 1 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( h )</td>
<td>2 ( \lceil m/2 \rceil^h )</td>
<td>2 ( \lceil m/2 \rceil^h ) ( \lceil m/2 \rceil - 1 )</td>
</tr>
</tbody>
</table>

The total number of keys then satisfies, at minimum:

\[
k = 1 + \sum_{i=0}^{h-1} 2 \lceil m/2 \rceil^i \left( \lceil m/2 \rceil - 1 \right)
\]

\[
= 1 + 2(\lceil m/2 \rceil - 1) \sum_{i=0}^{h-1} \lceil m/2 \rceil^i
\]

\[
= 1 + 2(\lceil m/2 \rceil - 1) \left( \lceil m/2 \rceil^h - 1 \right)
\]

\[
= 1 + 2(\lceil m/2 \rceil^h - 1)
\]

\[
= 2 \lceil m/2 \rceil^h - 1
\]
It follows that since this is a minimum that:

\[ k \geq 2 \lceil m/2 \rceil^h - 1 \]

Then:

\[
2 \lceil m/2 \rceil^h - 1 \leq k \\
\lceil m/2 \rceil^h \leq \frac{k + 1}{2} \\
h \leq \log_{\lceil m/2 \rceil} \left( \frac{k + 1}{2} \right)
\]

QED

**Example 3.1.** For example if a B-tree of order \( m = 20 \) contains \( k = 999 \) keys then the maximum possible height can be calculated via:

\[
h \leq \log_{\lceil 20/2 \rceil} \left( \frac{999 + 1}{2} \right) = \log_{10} 500 \approx 2.6987
\]

Since height must be an integer the maximum height is 2.

As an aside, the sparsest possible B-tree of order \( m = 20 \) with height \( h = 2 \) has:

\[ k = 2 \lceil 20/2 \rceil^2 - 1 = 199 \]

So our B-tree (with \( m = 20 \) and \( h = 2 \)) with 999 keys is far from being the sparsest.

To reinforce why \( h \leq 2 \) observe that the sparsest possible B-tree of order \( m = 20 \) with larger height \( h = 3 \) has:

\[ k = 2 \lceil 20/2 \rceil^3 - 1 = 1999 \]

Thus any B-tree of order \( m = 20 \) with larger height \( h = 3 \) has \( k \geq 1999 \) and so our 999 keys are not enough for a B-tree of order \( m = 20 \) with height \( h = 3 \).
**Theorem 3.0.2.** Suppose B-tree of order $m$ has $k$ keys and height $h$. Then:

(a) The densest possible such tree has:

$$k = m^{h+1} - 1$$

(b) Consequently any such tree has:

$$k \leq m^{h+1} - 1$$

(c) And it then follows that:

$$h \geq \log_m(k + 1) - 1$$

(d) Thus:

$$h = \Omega(\log k)$$

*Proof.* Omitted. Try it! It’s similar to but easier than the previous. \(\Box\)

**Example 3.2.** For example if a B-tree of order $m = 20$ contains $k = 999$ keys then the minimum possible height can be calculated via:

$$h \geq \log_{20}(999 + 1) - 1 \approx 1.3059$$

Since height must be an integer the minimum height is 2.

As an aside, the densest possible B-tree of order $m = 20$ with height $h = 2$ has:

$$k = 20^{2+1} - 1 = 7999$$

So our B-tree (with $m = 20$ and $h = 2$) with 999 keys is far from being the densest.

To reinforce why $h \geq 2$ observe that the densest possible B-tree of order $m = 20$ with smaller height $h = 1$ has:

$$k = 20^{1+1} - 1 = 399$$

Thus any B-tree of order $m = 20$ with smaller height $h = 1$ has $k \leq 399$ and so our 999 keys could not fit in a B-tree of order $m = 20$ with height $h = 1$.

**Theorem 3.0.3.** We have $h = \Theta(\log k)$.

*Proof.* Follows immediately. \(\Box\)
4 Advantages

There are several advantages to using B-trees, including:

- Due to the number of keys that a node may contain there is consequently less tree balancing required when inserts and deletions occur.
- When a key is found in a node, a large collection of close keys are immediately accessible. This manifests in file storage where access (finding the node) is far slower than reading the data (once the node has been found).
- When doing range queries (find all values between $x$ and $y$) it’s easy to pluck grouped values out of a node.
- Although as we’ll see the restructuring process must be managed carefully it turns out that it only happens infrequently because of the amount of empty key space permissible in a node.

5 Search

Since this is a multiway search tree, finding a key is easy, just like with a BST.

6 Tree Restructuring

6.1 Introduction

Recall that in a 2-3 tree we had to manage the issues of an overfull node when a 3-node became a 4-node, and an underfull node when a 2-node became a 1-node. Similarly for B trees we must manage:

- Overfull Node: A node has $m + 1$ children ($m$ keys).
- Underfull Node: A node has $\lceil m/2 \rceil - 1$ children ($\lceil m/2 \rceil - 2$ keys).

To deal with these issues we introduce three restructuring operations:

6.2 Rotation

The best possible situation arises when a node is underfull or overfull but there are extra keys in an adjacent sibling that we can use to restructure. This is called a key rotation and it’s the best possible because it’s not computationally intensive.

Suppose a key in node $n_1$ is overfull but the sibling $n_2$ directly to the right has key space. We take the largest key in $n_1$, promote it to and replace the next largest key in the parent which gets demoted to the $n_2$ sibling, putting it at the start of $n_2$’s keys. We also move $n_1$’s largest child to become $n_2$’s smallest child. This is a right rotation.
A mirror argument works if a node $n_1$ is overfull but the sibling $n_2$ directly to the left has key space. We take the smallest key in $n_1$, promote it to and replace the next smallest key in the parent which gets demoted to the $n_2$ sibling, putting it at the end of $n_2$’s keys. We also move $n_1$’s smallest child to become $n_2$’s largest child. This is a left rotation.

This approach will also work if a node is underfull and a sibling has an extra key it can donate.

Note that this will only work if there is an adjacent sibling in a position to help!

Here is an illustration of a right rotation in action for $m = 6$. The node on the left is overfull, it (temporarily) has seven children and six keys. Its sibling on the right has space so we rotate over.

6.3 Splitting

Consider a B-tree of order $m$. A node must at most have $m$ children and $m - 1$ keys but suppose it temporarily has $m + 1$ children and $m$ keys. This can arise directly from an insert or as part of the restructuring operation.

If the number of keys $m + 1$ is odd:

(a) Take the median key and promote it to the parent. This leaves an even $m - 1$ keys with $(m - 1)/2$ smaller and $(m - 2)/2$ larger.

(b) Take the $(m - 1)/2$ keys smaller than this median as well as the $(m - 1)/2 + 1 = (m + 1)/2$ leftmost children and create a new node.

(c) Take the $(m - 1)/2$ keys larger than this median as well as the $(m - 1)/2 + 1 = (m + 1)/2$ right children and create a new node.
To ensure this works we have to make sure that each of the new nodes is valid, meaning it has between ⌈m/2⌉ and m children. In other words we claim:

$$\left\lceil \frac{m}{2} \right\rceil \leq \frac{m + 1}{2} \leq m$$

But since m is odd we know m + 1 is even and so ⌈m/2⌉ = m/2 and this is obvious.

If the number of keys m is even:

(a) Take the lower median key and promote it to the parent. This leaves an odd m − 1 keys with (m − 2)/2 smaller and m/2 larger.

(b) Take the (m − 2)/2 keys smaller than this median as well as the (m − 2)/2 + 1 = m leftmost children and create a new node.

(c) Take the m/2 keys larger than this median as well as the m/2 + 1 = (m + 2)/2 right children and create a new node.

To ensure this works we have to make sure that each of the new nodes is valid, meaning it has between ⌈m/2⌉ and m children. In other words we first claim:

$$\left\lceil \frac{m}{2} \right\rceil \leq m \leq m$$

But since m is odd we know ⌈m/2⌉ = \frac{m+1}{2} and this is obvious.

We also claim:

$$\left\lceil \frac{m}{2} \right\rceil \leq \frac{m + 2}{2} \leq m$$

But since m is odd we know ⌈m/2⌉ = \frac{m+1}{2} and this is obvious.

Notice that as a result of this the parent gains a key and a child, meaning it may also overflow and we need to keep going up the tree to check.

**Note 6.3.1.** This seems convoluted but isn’t. Basically the overfull node will have m keys. The median (or lower median) key gets promoted and the left and right keys becomes the new nodes. Break up the children in the only sensible way.
Here is an illustration of a split in action for $m = 7$. In such a B-tree a non-leaf node may have between $\lceil m/2 \rceil = \lceil 7/2 \rceil = 4$ and $m = 7$ children and between 3 and 6 keys. The middle node is overfull, it (temporarily) has 8 children and 7 keys. We split it.

6.4 Merging

Consider a B-tree of order $m$. A node must at least have $\lceil m/2 \rceil$ children and $\lceil m/2 \rceil - 1$ keys but suppose it temporarily has $\lceil m/2 \rceil - 1$ children and $\lceil m/2 \rceil - 2$ keys. This can arise directly from a delete or as part of the restructuring operation.

First, if either of its adjacent siblings has more than $\lceil m/2 \rceil$ children then a key rotation does the job and we’re fine.

If not, then both of them have exactly $\lceil m/2 \rceil$ children. What we will do is pick one of them and merge the two together. Notice that when we do this the parent loses a child which means it must lose a key, so we will demote a key from the parent to the merged node. It turns out that this works quite well except for the fact that then the parent may be underfull now.

To verify that this all works out, observe that our underfull node has $\lceil m/2 \rceil - 1$ children and $\lceil m/2 \rceil - 2$ keys and the sibling we chose has at $\lceil m/2 \rceil$ children and $\lceil m/2 \rceil - 1$ keys. When we merge them there are $2 \lceil m/2 \rceil - 1$ children and $\lceil m/2 \rceil - 3$ keys. Note that this is not enough keys, but we’ll cross that bridge in a minute.

First we need to ascertain that:

$$\lceil m/2 \rceil \leq 2 \lceil m/2 \rceil - 1 \leq m$$

This can be proved with an even-odd argument as with splitting.
Now for the key issue. We’re short one key so we demote the key from the parent which was between the edges which connect the two adjacent nodes we are merging. The parent loses a key and a child, so its count is fine, but it might be underfull, and when this appears in the delete procedure it will just propagate.

Here is an illustration of a merge in action for $m = 6$. The left node is underfull but neither adjacent sibling can offer a key (left sibling not shown) so we merge with a sibling (the right in this case).
7 Insert

7.1 Algorithm

At this point insertion is easy. We find the correct leaf node and insert it and then we rotate and/or split up the tree until the restructuring is finished. Note that it’s possible that no restructuring is required at all.

Example 7.1. Let’s insert 19 into this B-tree:

![B-tree diagram]

Luckily there is space in its right sibling and so we can do a rotation:

Now we are done.
Example 7.2. Let's say we take the result of the previous exercise and insert 23. This overflows that leaf:

Neither adjacent sibling has space for a rotation so instead we split the overfull node in the middle, at the 22 and promote that middle 22. Unfortunately due to the promotion of the 22 the parent node is now overfull:

Lucky the parent has a sibling to the right which can accept a key, so we rotate it over:

Now we are done.

7.2 Time Complexity

Each rotation and split is $\Theta(1)$, occurring up to $\Theta(\log n)$ or $\Theta(\log k)$ times, for a total worst-case of $\Theta(\log n)$ or $\Theta(\log k)$. 

12
8 Delete

8.1 Algorithm

At this point deletion is easy. As with BST we assume we are removing a key from a leaf node, then we rotate and/or merge up the tree until the restructuring is finished. Note that it's possible that no restructuring is required at all.

Example 8.1. Let's delete 66 from this tree:

```
35  80
10  19  22  25
  2  15  21  24  29  41
  5  16  30  42
  7  17  31  47
```

The result yields an underfull node:

```
50  80
10  19  22  25
  2  15  21  24  29  41
  5  16  30  42
  7  17  31  47
```

Neither adjacent sibling can offer a key via rotation so the only choice is to merge with a sibling. Let's merge with the left sibling, which means that the sandwiched key 54 in the parent is pulled down:

```
50  80
10  19  22  25
  12  20  23  27
  2  15  21  24  29  41
  5  16  30  42
  7  17  31  47
```

Luckily the parent could give up a key with no issue and we are done.

8.2 Time Complexity

Each rotation and merge is $\Theta(1)$, occurring up to $\Theta(\lg n)$ or $\Theta(\lg k)$ times, for a total worst-case of $\Theta(\lg n)$ or $\Theta(\lg k)$.
9 B+ Trees

9.1 Structure

A B+ tree is a variation on a B-tree whereby:

- Internal nodes do not store values (the actual data) but rather just the keys. In our representations we haven't actually shown values, just keys, so the image below has some.
- All the key-value pairs are stored in the leaf nodes.
- Each leaf node has a pointer which points to the leaf node to the right.

Essentially the keys in the internal node are guideposts to the leaf nodes and the leaf nodes contain the key-value pairs which really constitute the data.

**Example 9.1.** Here is a B+ tree with \( m = 4 \). As with a B-tree the root node may have between 2 and \( m = 4 \) children and hence between 1 and 3 keys and every other node may have between \( \lceil m/2 \rceil = 2 \) and \( m = 4 \) children and hence between 1 and 3 keys: Each leaf node also has a value (some data) associated to its key.

![B+ Tree Diagram]

B+ trees have a few benefits including:

- Since the values (data) are saved only in the leaves this saves space in internal nodes.
- All queries (looking for keys with associated values) will reliably travel to the bottom of the tree.
- Range queries are especially nice. For example in the above tree if we’re looking for all keys (with values) in the range \([30, 88]\) we simply find the 30 and then follow the leaf nodes across.

9.2 Range Queries

**Example 9.2.** Same example with the above range query \([30, 88]\):

![Range Query Diagram]
9.3 Application: Databases

Imagine a sequential database in which each row has a non-unique ID and the rows are in no particular order.

If we wish to query for all rows with a particular ID we would need to go through all the rows sequentially and pick out those with that ID. This can take a long time.

If we create an index on the ID the result is typically a B+ tree in which the IDs are the keys and the values are the row numbers. Then when we search for a particular ID, or range of IDs, we quickly get back a list of rows we ought to look at and we can go directly those rows in the database to get the actual data.