Transpose of a Linear Transformation

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0.1 Introduction

These notes were just written to help me organize all the fiddly details.

0.2 Preliminaries

There are two things to recall:

N1: For a linear transformation $T : V \to ...$ we have:

$$\dim V = \dim(\operatorname{range} T) + \dim(\operatorname{null} T)$$

N2: For a subspace W of a vector space V we have:

$$\dim(V) = \dim(W) + \dim(W^0)$$

0.3 Definition of the Transpose

For a linear transformation $T : V \to W$ we can define a linear transform $T^t : W^* \to V^*$ called the *transpose* such that for $f \in W^*$ we define $T^t f \in V^*$ by $T^t f(\alpha) = f(T(\alpha))$.

Theorem:

The transformation T^t is linear.

Proof:

We claim that:

$$T^t(f + cg) = T^t f + cT^t g$$

These linear transformations are determined what they to do some $\alpha \in V$ and so observe that first:

$$T^t(f + cg)(\alpha) = (f + cg)(T\alpha) = f(T\alpha) + cg(T\alpha)$$

And second:

$$(T^t f + cT^t g)(\alpha) = T^t f(\alpha) + cT^t g(\alpha) = f(T\alpha) + cg(T\alpha)$$

0.4 Null, Rank, and Range of the Transpose

Theorem:

For a linear transformation $T : V \to W$ we have:

1. $\text{null}\,(T^t) = (\text{range}\,T)^0$
2. $\dim(\text{range}\,(T^t)) = \dim(\text{range}\,T)$
3. $\text{range}\,(T^t) = (\text{null}\,T)^0$

Proof:

For (a) note that:

$$f \in \text{null}\,(T^t) \iff T^t f = 0 \text{ (in } V^*) \iff \forall \alpha \in V, T^t f(\alpha) = 0 \iff \forall \alpha \in V, f(T\alpha) = 0 \iff f \in (\text{range}\,T)^0$$

For (b) note the following. Since $T^t : W^* \to V^*$ we have Fact I from N1:

$$\dim(W^*) = \dim(\text{null}\,(T^t)) + \dim(\text{range}\,(T^t))$$

Since $\text{range}\,T$ is a subspace of W we have Fact II from N2:

$$\dim(W) = \dim(\text{range}\,T) + \dim((\text{range}\,T)^0)$$

Now then, Fact I tells us:

$$\dim(\text{range}\,(T^t)) = \dim(W^*) - \dim(\text{null}\,(T^t))$$

Since $\dim(W^*) = \dim(W)$ and using (a) we get:

$$\dim(\text{range}\,(T^t)) = \dim(W) - \dim((\text{range}\,T)^0)$$

Now, Fact II tells us:

$$\dim(\text{range}\,(T^t)) = \dim(\text{range}\,T)$$

For (c) we first show that $\text{range}\,(T^t) \subseteq (\text{null}\,T)^0$ and then we show that the dimensions are the same.

Suppose $g \in \text{range}\,(T^t)$ so $\exists f \in W^*$ with $T^t f = g$. We claim that $g \in (\text{null}\,T)^0$, meaning we need to show that $g(\alpha) = 0$ for all $\alpha \in \text{null}\,T$. Let $\alpha \in \text{null}\,T$ so then $g(\alpha) = T^t f(\alpha) = f(T(\alpha)) = f(0) = 0$.

Since $T : V \to W$ we have Fact III from N1:

$$\dim V = \dim(\text{null}\,T) + \dim(\text{range}\,T)$$

Since $\text{null}\,T$ is a subspace of V we have Fact IV from N2:

$$\dim V = \dim(\text{null}\,T) + \dim((\text{null}\,T)^0)$$

Now then, observe that we have the following where the first equality is from (b), the second from Fact III, and the third from Fact IV:

$$\begin{aligned}\dim(\text{range}(T^t)) &= \dim(\text{range}\,T)\\ &= \dim V - \dim(\text{null}\,T)\\ &= \dim((\text{null}\,T)^0)\end{aligned}$$

0.5 Transformation, Transpose, and Matrices

Theorem:

Suppose $T : V \to W$ is a linear transformation and T^t is its transpose. Suppose $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ are bases for V and W respectively and $A' = \{f_1, ..., f_n\}$ and $B' = \{g_1, ..., g_n\}$ are their dual bases for V^* and W^* respectively.

Then the matrices $[T]_{B\leftarrow A}$ and $[T^t]_{A'\leftarrow B'}$ are matrix transposes of one another.

Proof:

To simplify let's denote these matrices by $[T]$ and $[T^t]$ respectively. By the definition and construction of these matrices, observe that for any $1 \leq i \leq n$ we have:

$$T\alpha_i = \sum_{k=1}^{m} [T]_{ki}\, \beta_k$$

And for any $1 \leq j \leq m$ we have:

$$T^t g_j = \sum_{k=1}^{n} \left[T^t\right]_{kj} f_k$$

Using the first of these, for any g_j and any α_i we have:

$$\begin{aligned} T^t g_j(\alpha_i) &= g_j(T\alpha_i) \\ &= g_j\left(\sum_{k=1}^{m} [T]_{ki}\, \beta_k\right) \\ &= \sum_{k=1}^{m} [T]_{ki}\, g_j(\beta_k) \\ &= [T]_{ji} \end{aligned}$$

Using the second of these, for any g_j and any α_i we have:

$$\begin{aligned} T^t g_j(\alpha_i) &= \sum_{k=1}^{n} \left[T^t\right]_{kj} f_k(\alpha_i) \\ &= \left[T^t\right]_{ij} \end{aligned}$$

Thus for any j and any i we have $[T]_{ji} = [T^t]_{ij}$ and we are done.