0.1 Introduction

These notes were just written to help me organize all the fiddly details.

0.2 Preliminaries

There are two things to recall:

N1: For a linear transformation $T : V \rightarrow \ldots$ we have:

$$\dim V = \dim (\text{range } T) + \dim (\text{null } T)$$

N2: For a subspace $W$ of a vector space $V$ we have:

$$\dim (V) = \dim (W) + \dim (W^0)$$
0.3 Definition of the Transpose

For a linear transformation $T : V \rightarrow W$ we can define a linear transform $T^t : W^* \rightarrow V^*$ called the transpose such that for $f \in W^*$ we define $T^t f \in V^*$ by $T^t f(\alpha) = f(T(\alpha))$.

Theorem:
The transformation $T^t$ is linear.

Proof:
We claim that:

$$T^t(f + cg) = T^t f + cT^t g$$

These linear transformations are determined what they do some $\alpha \in V$ and so observe that first:

$$T^t(f + cg)(\alpha) = (f + cg)(T\alpha) = f(T\alpha) + cg(T\alpha)$$

And second:

$$(T^t f + cT^t g)(\alpha) = T^t f(\alpha) + cT^t g(\alpha) = f(T\alpha) + cg(T\alpha)$$
### 0.4 Null, Rank, and Range of the Transpose

**Theorem:**
For a linear transformation $T : V \to W$ we have:

1. $\text{null}(T^t) = (\text{range } T)^0$
2. $\dim(\text{range }(T^t)) = \dim(\text{range } T)$
3. $\text{range }(T^t) = (\text{null } T)^0$

**Proof:**

For (a) note that:

$$f \in \text{null } (T^t) \iff T^t f = 0 \text{ (in } V^* \text{)} \iff \forall \alpha \in V, T^t f(\alpha) = 0 \iff \forall \alpha \in V, f(T\alpha) = 0 \iff f \in (\text{range } T)^0$$

For (b) note the following. Since $T^t : W^* \to V^*$ we have Fact I from N1:

$$\dim(W^*) = \dim(\text{null } (T^t)) + \dim(\text{range } (T^t))$$

Since range $T$ is a subspace of $W$ we have Fact II from N2:

$$\dim(W) = \dim(\text{range } T) + \dim((\text{range } T)^0)$$

Now then, Fact I tells us:

$$\dim(\text{range } (T^t)) = \dim(W^*) - \dim(\text{null } (T^t))$$

Since $\dim(W^*) = \dim(W)$ and using (a) we get:

$$\dim(\text{range } (T^t)) = \dim(W) - \dim((\text{range } T)^0)$$

Now, Fact II tells us:

$$\dim(\text{range } (T^t)) = \dim(\text{range } T)$$

For (c) we first show that range $(T^t) \subseteq (\text{null } T)^0$ and then we show that the dimensions are the same.

Suppose $g \in \text{range } (T^t)$ so $\exists f \in W^*$ with $T^t f = g$. We claim that $g \in (\text{null } T)^0$, meaning we need to show that $g(\alpha) = 0$ for all $\alpha \in \text{null } T$. Let $\alpha \in \text{null } T$ so then $g(\alpha) = T^t f(\alpha) = f(T(\alpha)) = f(0) = 0$.

Since $T : V \to W$ we have Fact III from N1:

$$\dim V = \dim(\text{null } T) + \dim(\text{range } T)$$

Since $\text{null } T$ is a subspace of $V$ we have Fact IV from N2:

$$\dim V = \dim(\text{null } T) + \dim((\text{null } T)^0)$$

Now then, observe that we have the following where the first equality is from (b), the second from Fact III, and the third from Fact IV:

$$\dim(\text{range } (T^t)) = \dim(\text{range } T)$$

$$= \dim V - \dim(\text{null } T)$$

$$= \dim((\text{null } T)^0)$$
0.5 Transformation, Transpose, and Matrices

Theorem:
Suppose \( T : V \to W \) is a linear transformation and \( T^t \) is its transpose. Suppose \( A = \{\alpha_1, \ldots, \alpha_n\} \) and \( B = \{\beta_1, \ldots, \beta_m\} \) are bases for \( V \) and \( W \) respectively and \( A' = \{f_1, \ldots, f_n\} \) and \( B' = \{g_1, \ldots, g_m\} \) are their dual bases for \( V^* \) and \( W^* \) respectively.

Then the matrices \( [T]_{B \leftrightarrow A} \) and \( [T^t]_{A' \leftrightarrow B'} \) are matrix transposes of one another.

Proof:
To simplify let’s denote these matrices by \( [T] \) and \( [T^t] \) respectively. By the definition and construction of these matrices, observe that for any \( 1 \leq i \leq n \) we have:

\[
T \alpha_i = \sum_{k=1}^{m} [T]_{ki} \beta_k
\]

And for any \( 1 \leq j \leq m \) we have:

\[
T^t g_j = \sum_{k=1}^{n} [T^t]_{kj} f_k
\]

Using the first of these, for any \( g_j \) and any \( \alpha_i \) we have:

\[
T^t g_j(\alpha_i) = g_j(T \alpha_i) = g_j \left( \sum_{k=1}^{m} [T]_{ki} \beta_k \right) = \sum_{k=1}^{m} [T]_{ki} g_j(\beta_k) = [T]_{ji}
\]

Using the second of these, for any \( g_j \) and any \( \alpha_i \) we have:

\[
T^t g_j(\alpha_i) = \sum_{k=1}^{n} [T^t]_{kj} f_k(\alpha_i) = [T^t]_{ij}
\]

Thus for any \( j \) and any \( i \) we have \( [T]_{ji} = [T^t]_{ij} \) and we are done.