MATH 115 Sections 2.7,2.8
Combining Functions, Inverses

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1. **Basic Combinations:** Suppose we're given two functions $f(x)$ and $g(x)$. Are there any ways we could combine them? For example we could add them. What would we call the new function? How about $f+g$? What's the rule? Well, $f+g$ takes x to $f(x)+g(x)$ so we'd write $(f+g)(x) = f(x) + g(x)$. We could also do the other basic operations, hence all together:

   - $(f+g)(x) = f(x) + g(x)$
   - $(f-g)(x) = f(x) - g(x)$
   - $(fg)(x) = f(x)g(x)$
   - $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

   Point out to the students that the left-hand side is just shorthand notation for the right-hand side. They like to think that $(f+g)(x)$ means $f+g$ *times* (x) and if they think this they get very confused.

   Example: Suppose $f(x) = x^2$ and $g(x) = x - 1$. Find all four of the above.

   Example: Suppose $f(x)$ and $g(x)$ are functions with $f(2) = 3$ and $g(2) = -7$. Find $(f+g)(2)$.

   Application: Suppose when a company produces x calculators its revenue is $R(x)$ and its cost is $C(x)$. The function $(R-C)(x) = R(x) - C(x)$ is the profit.

2. **Domains:** So what is the domain of these four? Well consider that anything you plug into $f+g$, for example, you must be able to plut into both f and g. Similarly for the other four so that the domain of these must be the *overlap* of the domains of f and of g. But for $\left(\frac{f}{g}\right)$ there's another catch. We also *cannot* have $g(x) = 0$.

   Example: Suppose $f(x) = \sqrt{x+5}$ and $g(x) = \sqrt{40-x}$. Find the domains of $f+g$, $f-g$ and fg. Note these are all the same. Then find the domain of $\left(\frac{f}{g}\right)$, noting that we must also eliminate where $g(x) = \sqrt{40-x} = 0$.

3. **Composition of Functions:** There's one last way we can combine functions. Consider the following example: Suppose if I work for x hours I produce $c(x)$ cookies, and suppose furthermore that if I produce x cookies I earn $p(x)$ profit. What would I earn from working 7 hours? What we really want is $p(c(7))$. Explain this. This leads to the following definition: Given $f(x)$ and $g(x)$:

   - The composition of f and g, denoted $f \circ g$, is the function with $(f \circ g)(x) = f(g(x))$.

   Example: Suppose $f(x) = \sqrt{2x+1}$ and $g(x) = |x| - 3$. Find $(f \circ g)(x)$.

   Example: Suppose $f(x)$ and $g(x)$ are functions with $f(2) = -1$, $f(5) = -3$, $g(-3) = 1$ and $g(5) = 2$. Find $(f \circ g)(5)$ and $(g \circ f)(5)$.

4. **Inverses:** Consider now the function $f(x) = 3x + 5$ and $g(x) = \frac{x-5}{3}$. Note that $f(0) = 5$ and $g(5) = 0$. Do a few more values. What's going on? We see that f and g do the reverse of one another. This makes sense when we think of f as "multiply by 3 then add 5" and g as "subtract 5 then divide by 3". We call these functions *inverses*. More formally we say $f(x)$ and $g(x)$ are inverses if $f(a) = b$ implies $g(b) = a$ and the reverse.

   Example: Suppose $f(x)$ and $g(x)$ are inverses and $f(2) = 7$. Then what can you conclude?

   If a function $f(x)$ has an inverse, usually we give it the special name $f^{-1}(x)$. Note that this is *not* a power, it's just notation. Thus if $f(a) = b$ then $f^{-1}(b) = a$.

   Example: So from earlier, if $f(x) = 3x + 5$ then $f^{-1}(x) = \frac{x-5}{3}$.

5. **Facts:** We need to know some things about inverses. Here they are:

   (a) Suppose $f(x)$ and $g(x)$ are given. How can we check if they're inverses? To be inverses means that if we do one then the other we get back where we started. With functions this means that $f(g(x)) = x$ but also $g(f(x)) = x$. Therefore if we want to know if f and g are inverses we simply check if both these things are true.

   Example: Check if $f(x) = \frac{x}{5} - 1$ and $g(x) = 5(x + 1)$ are inverses. (Yes)

   Example: Check if $f(x) = 2x - 3$ and $g(x) = \frac{x}{2} + 3$ are inverses. (No)

   (b) Suppose $f(x)$ is given and suppose you also know that $f(x)$ has an inverse. How can you find it? Simple steps:

      i. Replace $f(x)$ by y.

      ii. Interchange x and y.

      iii. Solve for y.

      iv. Replace y by $f^{-1}(x)$.

   Example: The function $f(x) = \sqrt[3]{2x + 1}$ has an inverse. Find it.

   (c) Suppose the graph of $f(x)$ is given and we want to only know *if* it has an inverse. How do we know? The *horizontal line test* says that if we can draw a line which hits the function more than once then it does not have an inverse.

   If you have time you can explain why this works with a specific graph. If, for example, the line hits $(2, 5)$ and $(7, 5)$ then $f(2) = 5$ and $f(7) = 5$ but then what would $f^{-1}(5)$ be?

   Example: Draw some with and some without.

   (d) Suppose the graph of $f(x)$ is given and you want to draw the inverse. How can you? Easy, the graphs of $f(x)$ and $f^{-1}(x)$ are reflections of one another over the diagonal mirror $y = x$.

   If you have time you can explain why this works with a specific graph. If $(2, 5)$ is on the graph then $f(2) = 5$ so $f^{-1}(5) = 2$ so $(5, 2)$ is on the graph of the inverse. Doing a few such points with a function makes it very clear.

   Example: Draw some.