

MATH 115 Sections 3.1b Lecture Notes
Rational Functions - Part 1 of 2

1. **Introduction:** We are going to analyze rational functions in two steps. Today we're simply going to look at a few features of the function and then on Friday we'll see how to use these features to draw a nice graph.

2. **Definition:** A *rational function* is a function of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Examples: $f(x) = \frac{3x+6}{x-10}$, $f(x) = \frac{2}{(x+7)(x-1)}$, $f(x) = \frac{x^2+x}{3-x}$, $f(x) = \frac{x^2+6x+5}{x^2-9}$, $f(x) = \frac{1}{x}$

3. **Features:** Let's look at some obvious features first.

- *y*-intercept. As with every function ever, we find the *y*-intercept by plugging in $x = 0$.
 Example: Among others, do something like $f(x) = \frac{x+3}{x^2-x}$ since it has no *y*-intercept.
- *x*-intercepts. As with every function ever, we find the *x*-intercepts by setting the whole thing equal to 0. But since a fraction only = 0 if the numerator does, we can just set the numerator = 0 and solve.
 Example: Among others, do something like $f(x) = \frac{x^2+1}{x^2-5x-6}$ since it has no *x*-intercepts.

Additional useful note the book doesn't mention: The behavior of the graph at the *x*-intercepts replicates that for polynomial functions. In other words, the multiplicity of the intercept dictates whether it cuts hard, soft or turns. It's okay for the students to use this so you might want to make it part of your lecture.

4. **Domain and Vertical Asymptotes:** What is the domain of a rational function? Well we've done this already - we simply need to see where the denominator = 0.

Simple example: $f(x) = \frac{1}{x-3}$

To find the domain we note that $x - 3 = 0$ is bad and so we can't have $x = 3$. Hence the domain is $(-\infty, 3) \cup (3, \infty)$.

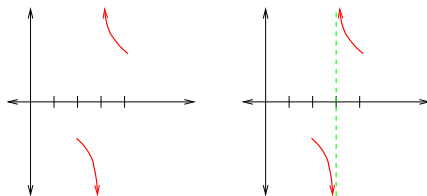
But the real issue here is this: What happens near that *x* value? In other words, as *x* gets close to $x = 3$, what does the function do? To see this let's look at the example above and plug in some points. We'll look at *x*s close to $x = 3$, some higher and some lower:

<i>x</i>	$f(x)$
2	$\frac{1}{2-3} = -1$
2.9	$\frac{1}{2.9-3} = -10$
2.99	$\frac{1}{2.99-3} = -100$
3	undefined
3.01	$\frac{1}{3.01-3} = 100$
3.1	$\frac{1}{3.1-3} = 10$
4	$\frac{1}{4-3} = 1$

We see as *x* gets closer to 3 from the left ($x < 3$) that $f(x)$ gets more and more negative; it heads to $-\infty$. To abbreviate this we write "As $x \rightarrow 3^-$ we have $f(x) \rightarrow -\infty$ ".

We also see as *x* gets closer to 3 from the right ($x > 3$) that $f(x)$ gets more and more positive; it heads to ∞ . To abbreviate this we write "As $x \rightarrow 3^+$ we have $f(x) \rightarrow \infty$ ".

In the graph we see this. The picture on the right has a guideline drawn in. This is the line $x = 3$.



Definition: We call the line $x = 3$ a *vertical asymptote* for the graph.

In general then how do we find vertical asymptotes? Well it looks like we just see where the denominator is $= 0$ and that's 99% true. The one issue is that when confronted with a rational function we need to factor the numerator and denominator and cancel like terms before making this call. For us this rarely happens but is worth mentioning.

Example: If $f(x) = \frac{x^2+5x}{x^2-2x-35}$ then we rewrite as $f(x) = \frac{x(x+5)}{(x+5)(x-7)} = \frac{x}{x-7}$ so $x = 7$ is the only VA.

Note to instructors: We gloss over the issue of holes in graphs. In reality there is only maybe one WebAssign homework with a canceling term and they don't need to graph it.

Example: If $f(x) = \frac{x^2+7x+6}{x^2-9} = \frac{(x+1)(x+6)}{(x-3)(x+3)}$ the VAs are $x = 3$ and $x = -3$.

The best way to analyze what happens *at* each VA is to simply plug in an x -value on each side of the VA and see if the function is going $+$ (to ∞) or $-$ (to $-\infty$). As a shortcut we can plug it into the factored version and only look at the sign of the terms:

Example: For the $f(x)$ above we have two VAs with behavior:

At the VA $x = -3$:

Left side $x = -3.1$: $f(-3.1) = \frac{(-3.1+1)(-3.1+6)}{(-3.1-3)(-3.1+3)} = \frac{(-)(+)}{(-)(-)} = -$ so it goes down.

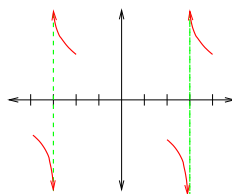
Right side $x = -2.9$: $f(-2.9) = \frac{(-2.9+1)(-2.9+6)}{(-2.9-3)(-2.9+3)} = \frac{(-)(+)}{(-)(+)} = +$ so it goes up.

At the VA $x = 3$:

Left side $x = 2.9$: $f(2.9) = \frac{(2.9+1)(2.9+6)}{(2.9-3)(2.9+3)} = \frac{(+)(+)}{(-)(+)} = -$ so it goes down.

Right side $x = 3.1$: $f(3.1) = \frac{(3.1+1)(3.1+6)}{(3.1-3)(3.1+3)} = \frac{(+)(+)}{(+)(+)} = +$ so it goes up.

A partial sketch would then be:



Note: Once we've got the last features of the graph and once we're actually going to sketch, it's not always necessary to plug lots of values in. We'll clarify this as the situation arises.