

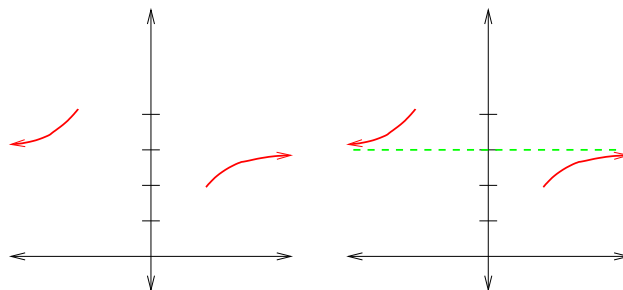
MATH 115 Sections 3.6b Lecture Notes
Rational Functions - Part 2 of 2

- 1. Introduction:** On Wednesday we looked at intercepts and vertical asymptotes of rational functions. This is very much the “middle behavior” of these functions - it is what goes on in the center of the graph. Today we wish to address what happens to the far left and far right, much like we did with polynomials although the method is completely different.
- 2. The Rule, an Example:** There are three possibilities for the left and right behavior of rational functions and they are easy to analyze. Before giving a general rule we’ll examine one single example:

Example: Let $f(x) = \frac{6x}{2x+1}$. For now all we can do is plug in some points to the far left and right. We do so and get the following, written left-to-right for easy understanding:

x	-1000	-100	-10	10	100	1000
$f(x) \approx$	3.001	3.015	3.16	2.86	2.99	2.999

We see that as $x \rightarrow \infty$, $f(x) \rightarrow 3$. Moreover as it goes left it goes down to 3 and as it goes right it goes up to 3. Thus the picture will look like this. The picture on the right has a guideline added. This is the line $y = 3$.



Definition: This horizontal line is called a *horizontal asymptote* (HA).

Where does the 3 come from? The only obvious place is from the 6 over the 2, and this is one of the three cases in the rule:

- 3. The Rule:** Suppose $\frac{p(x)}{q(x)}$ is a rational function. There are three cases:
 - (a) If the degree of $p(x) =$ degree of $q(x)$ then we divide the leading coefficients. In other words, $y = \frac{\text{LC of } p(x)}{\text{LC of } q(x)}$ is the HA.
 - (b) If the degree of $p(x) <$ degree of $q(x)$ then $y = 0$ is the HA.
 - (c) If the degree of $p(x) >$ degree of $q(x)$ then there is no horizontal asymptote. Sometimes there is a *slant asymptote* but we never need to sketch these. Basically the graph goes off to $\pm\infty$.

Example: Determine the HA (if any) for the following:

$$a(x) = \frac{2x^2+3x+1}{5x^2-3},$$

$$b(x) = \frac{6x-1}{x^2+2x+1},$$

$$c(x) = \frac{6x^2+7x+1}{x-3},$$

$$d(x) = \frac{2(x-3)(x+4)^2}{(5x-1)^2(x+6)}.$$

Note that for the fourth one the degree and leading coefficient are not obvious. We might need to do our polynomial trick to find the leading term of each one, dropping all but the LT of each factor.

4. Graphing:

We can now put it all together to draw some graphs. For each of the following use all the behavior we've analyzed. Only plot additional points if you can't determine the behavior without them.

Example: $f(x) = \frac{(x-1)(x+3)}{(2x+4)(x-7)}$. This entire graph can be drawn without additional points so it's a very good one to discuss with them about why. As long as you use all the data we've looked at, including the behavior (cuts hard!) at the x - intercepts, everything is straightforward.

Example: $g(x) = \frac{x-3}{x^2+6x+5}$. This graph can be partially drawn but to finish it requires two additional points. Make sure they graph what they can and then understand what's yet to be done and how two points will help.

Example: $h(x) = \frac{(x-1)(x+3)}{x}$. This definitely needs points to get anywhere at all!