MATH 115 Sections 4.1 Lecture Notes
Exponential Functions

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1. **Introduction:** Basically this section of the course (chapters 3 and 4) cover the graphs of some basic functions. We've dealt with polynomial and rational functions, now we need to look at exponential and logarithmic functions. Exponential functions are important because they come up frequently: population growth, radioactive decay, measurement of sound and earthquake intensity, and so on.

2. **Basic Definition:** The *exponential function with base* a is the function $f(x) = a^x$ with $a > 0$ and $a \neq 1$.

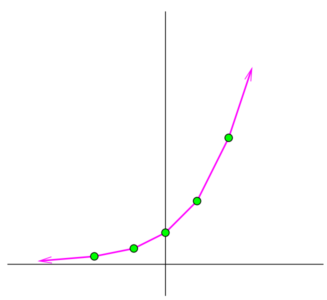
   Examples: $f(x) = 2^x$, $g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = e^x$.

   Generally an *exponential function* can be any transformation of one of these.

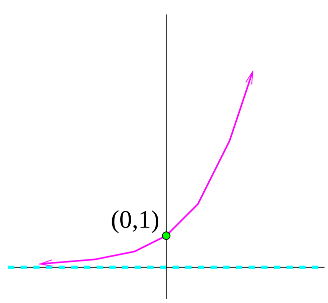
   Note that $a > 0$ because if we had something like $f(x) = (-2)^x$ then this is undefined for lots of values of x like $f\left(\frac{1}{2}\right) = \sqrt{-2}$. Also $a \neq 1$ because then we'd have $f(x) = 1^x = 1$ which is pretty boring.

3. **Basic Graphs:** Let's take a look at two representative exponential graphs, $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$. Since we have no idea what these looks like, let's plot points:

   First, $f(x) = 2^x$. If we plot points for $x = -2, -1, 0, 1, 2$ we get the graph:



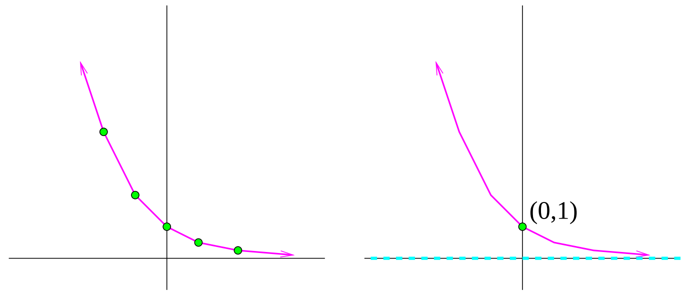
   In fact, if we replace 2^x by a^x for $a > 1$, we still keep this same basic shape. What do we mean by this? The graph of $f(x) = a^x$ for $a > 1$ looks like:



   Note the important features:

   - Increasing.
   - Passes through $(0, 1)$ (because $f(0) = a^0 = 1$ for any a.)
   - Horizontal asymptote at $y = 0$.
   - Domain $(-\infty, \infty)$ and range $(0, \infty)$.

Likewise we can do the same for $f(x) = \left(\frac{1}{2}\right)^x$. Plotting $x = -2, -1, 0, 1, 2$ we get a similar shape which is the same if $\frac{1}{2}$ is replaced by any other $0 < a < 1$,



Again the important features:

- Decreasing.
- Passes through $(0, 1)$ (because $f(0) = a^0 = 1$ for any a.)
- Horizontal asymptote at $y = 0$.
- Domain $(-\infty, \infty)$ and range $(0, \infty)$.

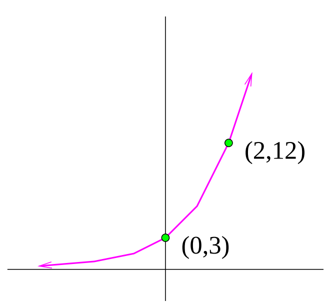
4. **Transformations:** We can now apply transformations to these just like we have for any function. Note that most of the features listed above can change once transformations have taken place:

   Example: Sketch $f(x) = 2^{x-3}$. Note that this is 2^x with x replaced by $x - 3$. This means it's a shift right by 3 units. If we sketch it (do so!) we see that $(0, 1)$ has moved to $(3, 1)$ but the asymptote has not moved. The domain and range have not changed either.

   Example: Sketch $f(x) = 10 - 10^x$. Note that this is the same as $f(x) = -10^x + 10$. Also note that the $-$ is **not** part of the base. That is, it's $-(10^x) + 10$ and not $(-10)^x + 10$ due to order of operations. Therefore this graph is 10^x reflected in the x-axis and then shifted up by 10 units. If we sketch it (do so!) we see that $(0, 1)$ flips to $(0, -1)$ and then shifts to $(0, 9)$. The asymptote flips (no effect!) and then shifts to $y = 10$. The graph is now decreasing. The domain is still $(-\infty, \infty)$ but the range has changed to $(-\infty, 10)$.

   Example: Sketch $f(x) = 2\left(\frac{1}{3}\right)^{x+1}$. Note that this is $\left(\frac{1}{3}\right)^x$ stretched by a factor of 2 and then shifted left by 1 unit. The stretching moves $(0, 1)$ to $(0, 2)$ and the shifting moves it to $(-1, 2)$. The asymptote stays the same.

   Example: Suppose an exponential graph looks like the picture below and has the form $f(x) = Ca^x$.



   Can we find this function: Sure, since it passes through $(0, 3)$ we know $3 = f(0) = Ca^0 = C$ and so $f(x) = 3a^x$. Then since it passes through $(2, 12)$ we know $12 = f(2) = 3a^2$ and so $a^2 = 4$. Since negative bases are not possible we must have $a = 2$ and so $f(x) = 3(2)^x$. Note that **this is not** 6^x due to order of operations.