MATH 115 Sections 4.3 Lecture Notes
Rules/Laws of Logarithms

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1. **Introduction:** Just as exponentials have properties (like $x^2x^3 = x^5$ and $\frac{2^x}{2^y} = 2^{x-y}$, logarithms also have useful properties. We'd like to list the four useful properties and see what they're good for.

2. **Three Properties:** We will focus on three main properties first.

   (a) **Product Rule:** $\log_a(MN) = \log_a M + \log_a N$

   (b) **Quotient Rule:** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

   (c) **Power Rule:** $\log_a M^P = P \log_a M$

   There are a number of ways to implement these. Some of these will appear more useful to you right now than others, but all will be useful in time.

Example: Evaluate $\log 5 + \log 20$.

Solution: Neither is particularly nice but $\log 5 + \log 20 = \log(5 \cdot 20) = \log 100 = 2$. Note we used the product rule in reverse here.

Example: Compress $\log 13 + \frac{1}{2}\log 5 - \log 7$ to the logarithm of a single number.

Solution: $\log 13 + \frac{1}{2}\log 5 - \log 7 = \log 13 + \log 5^{1/2} - \log 7 = \log(13\sqrt{5}) - \log 7 = \log\left(\frac{15\sqrt{5}}{7}\right)$.

Example: Break down and simplify $\log_2\left(\frac{3}{8}\right)$ as much as possible.

Solution: We can do $\log_2\left(\frac{5}{8}\right) = \log_2(5) - \log_2(8) = \log_2(3) - 3$. This may seem useless but if you knew somehow that $\log_2(3) \approx 1.58$ then you'd know that $\log_2\left(\frac{3}{8}\right) \approx 1.58 - 3 = -1.42$.

Example: Expand the logarithmic expression $\log_3\left(\frac{x}{5}\right)$.

Solution: We see $\log_3\left(\frac{x}{5}\right) = \log_3 x - \log_3 5$. Note that WebAssign usually tells you that in your answer you should replace $r = \log_3 x$ and $s = \log_3 5$ so the answer would be $r - s$ here.

Example: Using $r = \log_2 x$ and $s = \log_2 y$, rewrite $\log_2(8xy)^8$ in terms of r and s.

Solution: $\log_2(8xy)^8 = 8\log_2(8xy) = 8\left(\log_2 8 + \log_2 x + \log_2 y\right) = 8\left(3 + r + x\right)$.

Example: Using $r = \ln a$, $s = \ln b$ and $t = \ln c$, rewrite $\ln\left(\frac{a^9}{b^3\sqrt{c}}\right)$ in terms of r, s and t.

Solution: $\ln\left(\frac{a^9}{b^3\sqrt{c}}\right) = \ln(a^9) - \ln(b^3c^{1/2}) = 9\ln a - \left(\ln b^3 + \ln c^{1/2}\right) = 9\ln a - \left(3\ln b + \frac{1}{2}\ln c\right) = 9r - \left(3s + \frac{1}{2}c\right)$.

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3. **Fourth Property:** A fourth property is very useful when you need to find something like $\log_2(3)$ on your calculator. Your calculator only has ln and log. (In truth there is probably a $\log_x y$ buried in a menu somwhere.)

   (d) **Change of Base:** $\log_a b = \frac{\log_c b}{\log_c a}$ for any c you like.

Example: Suppose I want to know $\log_2 3$.

Solution: This rule tells me that $\log_2 3 = \frac{\log_5 3}{\log_5 2} = \frac{\log_3 3}{\log_3 2} = \frac{\log_8 3}{\log_8 2} = \frac{\log 3}{\log 2} = \frac{\ln 3}{\ln 2} = ...$ for any base we like. Of course the last two are the most useful since we can do them on the calculator. Your choice, it does not matter. $\log_2 3 = \frac{\ln 3}{\ln 2} \approx 1.584962501$.

Example: This also gives us a really sneaky way to evaluate things like $\log_8 4$, where we know they're both powers of 2 but doing it in our heads gives us a headache.

Solution: $\log_8 4 = \frac{\log_2 4}{\log_2 8} = \frac{2}{3}$.

Example: How about $\log_{1/9} 3$?

Solution: $\log_{1/9} 3 = \frac{\log_3 3}{\log_3 \frac{1}{9}} = \frac{1}{-2} = -\frac{1}{2}$.
