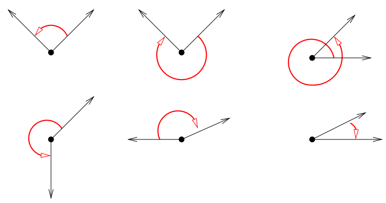
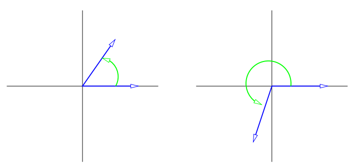
MATH 115 Section 6.1 Lecture Notes

1. **Introduction:** While most of the students are familiar with angles, intuitively, usually many of them do not understand the actual formal definition of an angle or how an angle is measured.

2. **Definition:** An *angle* is composed of: two rays (half-lines) meeting at a point called a vertex. One of the rays is the *initial side* and the other is the *terminal side*. We imagine that the initial side rotates around the vertex to the terminal side. We must give the manner of rotation (which direction and how far). Here are some angles. All are different! The direction of rotation also tells us which side is the initial side and which is the terminal side.



3. **Standard Position:** We say an angle is in *standard position* if its vertex is at the origin in the xy-plane and its initial side lies along the positive x-axis. We can imagine picking any of the above angles up and repositioning them in this way. For example, here are some angles in standard position.



4. **Measuring and Angle:** In order to *measure* an angle, we (formally) follow the steps:

   (a) Place the angle in standard position.

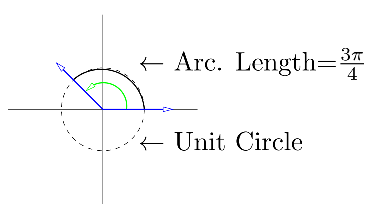
   (b) Overlay a unit circle.

   (c) Look at the portion of the unit circle contained within the angle. (We say the arc subtends the angle.)

   (d) Measure the length of this arc. If the angle has multiple rotations then we count the distance multiple times. If the angle rotates clockwise then we count the distance as negative.

   This gives us the measure of the angle in *radians*. This is the best way to measure an angle because it doesn't come from any outside source. This is the way the students should be most comfortable because this is the way that angles are measured in most of their calculus classes.

Example: The following angle has measure $\frac{3\pi}{4}$ radians, or $\frac{3\pi}{4}$ rad.



We can also measure the angle in *degrees*, where one full rotation is 360 degrees. We then have the conversion:

(a) Radians to Degrees: Multiply by $\frac{180}{\pi}$

(b) Degrees to Radians: Multiply by $\frac{\pi}{180}$

The angle measured above is then $\frac{3\pi}{4} \cdot \frac{180}{\pi} = 135$ degrees or $135°$.

Example: Do some more conversions, including some of the classics like $\frac{\pi}{6}$ and so on.

IMPORTANT: If we discuss an angle with no units given we mean radians. For degrees we must specifically write degrees or $°$.

5. **Coterminal Angles:** Two angles are said to be *coterminal* if, when placed in standard position, their terminal sides are the same. More easily, this occurs when the angles differ by a multiple of $2\pi = 180°$.

Example: Find several angles coterminal to $\frac{\pi}{6}$.

Example: Find an angle in $[0, 360)$ which is coterminal to $-530°$.

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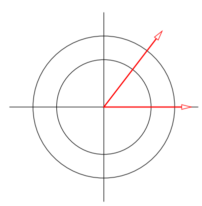
6. **Length of an Arc:** Suppose instead of being placed on a unit circle, an angle of θ is placed on a circle of radius r. What is the length s of the arc it contains? In other words, what length of arc subtends that angle? In the following picture the interior circle is the unit circle and the exterior circle has radius r.



The length of the arc on the unit circle is θ and then by similarity (arc length:radius) we have $\frac{\theta}{1} = \frac{s}{r}$ and so $s = r\theta$.

Example: What is the length of the arc which subtends an angle of $\frac{\pi}{6}$ on a circle of radius 10?

Example: Suppose an arc of length 5 subtends an angle of $80°$ on a circle of radius 7. What is the radius of the circle?
