MATH 115 Section 7.5 Lecture Notes

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1. **Definition:** Trigonometric functions are often used to model real-world situations. For example, sine functions model the motion of springs and pendulums and of electromagnetic waves.

   For this reason it is often necessary to solve a trigonometric equation. A *trigonometric equation* is an equation in which the desired variable is inside one or more trig functions.

   Examples: $\sin x = \frac{1}{2}$, $\sin x + \cos x = 0.4$, $\sin x \cos x - \cos x = 0$, $\sin(2x) - \sin x = 1$

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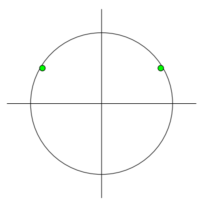
2. **Stumbling Through an Example:** Let's first look at a very basic trig equation:

   Example: $\sin x = \frac{1}{2}$.

   When we see this we immediately know that a possibility is $x = \frac{\pi}{6}$. But since sine repeats every 2π, we could have $x = \frac{\pi}{6} + 2\pi$, $x = \frac{\pi}{6} + 4\pi$, $x = \frac{\pi}{6} + 6\pi$, and so on, or $x = \frac{\pi}{6} - 2\pi$ and so on. To encapsulate these all in one expression, we write $x = \frac{\pi}{6} + 2\pi k$, where k is any integer.

   But wait, there's more! We could also have $x = \frac{5\pi}{6}$ and logically therefore $x = \frac{5\pi}{6} + 2\pi k$.

   Where y-value$=\sin x = \frac{1}{2}$

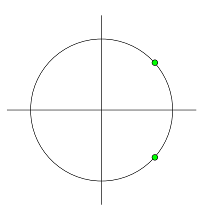
   In summary:

   Solutions in $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

   Solutions overall: $x = \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$.

3. **General Approach with Sine and Cosine:** For a simple equation involving a single sine or cosine on one side, first find the solutions in the interval $[0, 2\pi)$. There might be 2, 1 or 0 solutions. Then add $2\pi k$ to each.

   Example: Solve $\cos x = \frac{\sqrt{2}}{2}$. We know that $x = \frac{\pi}{4}$ and $x = \frac{7\pi}{4}$ work, so then in total we have $x = \frac{\pi}{4} + 2\pi k$ and $x = \frac{4\pi}{4} + 2\pi k$ where k could be any integer.

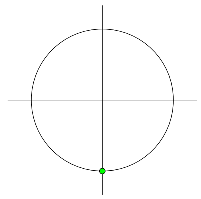
   Where x-value$=\cos x = \frac{\sqrt{2}}{2}$

   In summary:

   Solutions in $[0, 2\pi)$: $x = \frac{\pi}{4}, \frac{7\pi}{4}$.

   Solutions overall: $x = \frac{\pi}{4} + 2\pi k, \frac{7\pi}{4} + 2\pi k$.

Example: Solve $\sin x = -1$. We know $x = \frac{3\pi}{2}$ works, so then in total we have $x = \frac{3\pi}{2} + 2\pi k$ where k could be any integer.



Where y-value=$\sin x = -1$

In summary:

Solution in $[0, 2\pi)$: $x = \frac{3\pi}{2}$.

Solutions overall: $x = \frac{3\pi}{2} + 2\pi k$.

Example: Solve $\cos x = 3$. No value of x works because cosine is the x-value and this is always between -1 and 1.

In summary:

Solution in $[0, 2\pi)$: None.

Solutions overall: None.

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4. **General Approach with Tangent:** For a simple equation involving tangent, things are different. Since tangent repeats every π units, first find the solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and then add πk where k is any integer.

   IMPORTANT: Sometimes we also want to know which solutions are in $[0, 2\pi)$ so we'll list those too.

Example: Solve $\tan x = \frac{1}{\sqrt{3}}$. We know $x = \frac{\pi}{6}$ works, so then $x = \frac{\pi}{6} + \pi k$ gets them all.

In summary:

Solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$: $x = \frac{\pi}{6}$.

Solutions in $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{7\pi}{6}$.

Solutions overall: $x = \frac{\pi}{6} + \pi k$.

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5. **General Approach with Anything Else:** For a simple equation involving any of cotangent, secant or cosecant, it's best if we can first convert to sine, cosine or tangent. This doesn't always work smoothly though; we take care of the exceptions as they arise.

Example: Solve $\sec x = 2$. First observe that $\cos x = \frac{1}{2}$ and so $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ work, so then $x = \frac{\pi}{3} + 2\pi k$ and $x = \frac{5\pi}{3} + 2\pi k$.

In summary:

Solutions in $[0, 2\pi)$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

Solutions overall: $x = \frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k$.

Example: Solve $\cot x = 0$. If we convert to tangent we get $\tan x = \frac{1}{0}$ =undefined. This occurs at $x = \frac{\pi}{2}$ (where tangent has an asymptote), so then $x = \frac{\pi}{2} + \pi k$ gets them all.

In summary:

Solution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$: $x = \frac{\pi}{2}$.

Solutions in $[0, 2\pi)$: $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Solutions overall $x = \frac{\pi}{2} + \pi k$.

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6. **Final Type:** If the equation involves more than one trig expression then we may need to do some basic algebra first. Generally this means factoring and setting things equal to 0. Sometimes it involves rewriting the equation using a trig identity first. Here are some examples:

Example: Solve $\sin x \cos x = 0$.

We must have $\sin x = 0$ or $\cos x = 0$. These can then be solved independently like before.

Example: Solve $\cos x - \sin^2 x = 1$.

Here we change $\sin^2 x$ to $1 - \cos^2 x$, cancel the 1s and factor.

Example: Solve $2\sqrt{3} - \sqrt{3}\sin x + 4\cos x - 2\sin x \cos x = 0$.

Here we must factor.

Example: Solve $3 + \tan^2 x = 0$.

Example: Solve $\sin(2x) = \cos x$.

Here the best move is to use the double-angle formula for sine to rewrite this as $2\sin x \cos x = \cos x$. At this point the students often make the mistake of cancelling $\cos x$ but this loses solutions. Instead, emphasize that they should move everything to one side and factor.

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7. **Note:** When doing the problems above, make sure you clearly summarize which solutions are in $[0, 2\pi)$ and make sure you list them in increasing order. This is how WebAssign demands them and so it's best if you do it this way.