Math 241 Fall 2014 Final Exam Solutions

1. Given the line $L$ with symmetric equation $x = \frac{y - 1}{3} = \frac{z}{2}$, the plane with equation $P$ given by $9x - 2y - z = 0$ and the point $Q = (1, -2, 5)$:

   (a) Determine whether the line $L$ is parallel to the plane $P$. [10 pts]

   **Solution:**
   
   The direction vector for $L$ is $\vec{L} = 1\hat{i} + 3\hat{j} + 2\hat{k}$.
   The normal vector for $P$ is $\vec{N} = 9\hat{i} - 2\hat{j} - 1\hat{k}$.
   
   We check $\vec{L} \cdot \vec{N} = (1)(9) + (3)(-2) + (2)(-1) = 1 \neq 0$ so the line is not parallel to the plane.

   (b) Find the distance from the point $Q$ to the line $L$. Simplify. [15 pts]

   **Solution:**
   
   Pick a point on the line: $R = (0, 1, 0)$.
   
   Then
   
   $\overrightarrow{RQ} = 1\hat{i} - 3\hat{j} + 5\hat{k}$
   
   and so
   
   $\text{dist} = \frac{||\overrightarrow{RQ} \times \vec{L}||}{||\vec{L}||} = \frac{||-21\hat{i} + 3\hat{j} + 6\hat{k}||}{||1\hat{i} + 3\hat{j} + 2\hat{k}||} = \frac{\sqrt{(-21)^2 + 3^2 + 6^2}}{\sqrt{1^2 + 3^2 + 2^2}}$
2. Let the position of an object in motion be given by $\vec{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} + e^t \mathbf{k}$.

(a) Find the velocity and acceleration of the object at any $t$. \[10 \text{ pts}\]

**Solution:**

We calculate

$$\vec{r}'(t) = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j} + e^t \mathbf{k}$$

and

$$\vec{r}''(t) = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t) \mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t) \mathbf{j} + e^t \mathbf{k}$$

$$= -2e^t \sin t \mathbf{i} + 2e^t \cos t \mathbf{j} + e^t \mathbf{k}$$

(b) Write down the integral for the distance traveled by the object between $t = 0$ and $t = 2$ but do not evaluate. \[5 \text{ pts}\]

**Solution:**

We have

$$||\vec{r}'(t)|| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2}$$

so that the length is

$$\int_0^2 \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2} \, dt$$

(c) Compute the curvature of the object’s path at $t = 0$. \[10 \text{ pts}\]

**Solution:**

We have

$$\vec{r}'(0) = 1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k}$$

and

$$\vec{r}''(0) = 0 \mathbf{i} + 2 \mathbf{j} + 1 \mathbf{k}$$

so that

$$\kappa = \frac{||(1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k}) \times (0 \mathbf{i} + 2 \mathbf{j} + 1 \mathbf{k})||}{||1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k}||^3} = \frac{||-1 \mathbf{i} - 1 \mathbf{j} + 2 \mathbf{k}||}{||1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k}||^3} = \frac{\sqrt{6}}{3^{3/2}}$$
3. Use the method of Lagrange Multipliers to determine the maximum and minimum values of the function \( f(x, y) = xy \) subject to the constraint \( 4x^2 + y^2 = 4 \). You may assume that the maximum and minimum exist.

**Solution:**

We assign \( g(x, y) = 4x^2 + y^2 \) and then we set up the system:

\[
\begin{align*}
  y &= \lambda(8x) \tag{1} \\
  x &= \lambda(2y) \tag{2} \\
  4x^2 + y^2 &= 4 \tag{3}
\end{align*}
\]

Equation (1) tells us that \( \lambda = \frac{y}{8x} \) unless \( x = 0 \) (but \( x = 0 \) would give us \( y = 0 \) in (1) and this contradicts (3)).

Equation (2) tells us that \( \lambda = \frac{x}{2y} \) unless \( y = 0 \) (but \( y = 0 \) would give us \( x = 0 \) in (1) and this contradicts (3)).

Thus \( \frac{y}{8x} = \frac{x}{2y} \) and so \( 2y^2 = 8x^2 \) or \( y^2 = 4x^2 \).

Plugging this into (3) yields \( 8x^2 = 4 \) so \( x = \pm \sqrt{1/2} \).

If \( x = \sqrt{1/2} \) then (3) tells us \( x = \pm \sqrt{2} \) and the same for the other \( x \).

Thus we have four points: \((-\sqrt{1/2}, -\sqrt{2}), (-\sqrt{1/2}, +\sqrt{2}), (+\sqrt{1/2}, -\sqrt{2}), \) and \((+\sqrt{1/2}, +\sqrt{2})\)

Next:

\[
\begin{align*}
  f(-\sqrt{1/2}, -\sqrt{2}) &= 1 \\
  f(-\sqrt{1/2}, +\sqrt{2}) &= -1 \\
  f(+\sqrt{1/2}, -\sqrt{2}) &= -1 \\
  f(+\sqrt{1/2}, +\sqrt{2}) &= 1
\end{align*}
\]

Thus the minimum is \(-1\) and the maximum is \(1\).
4. (a) Let \( D(x, y) = 300 - 2x^2 - 3y^2 \) denote the depth of a lake in feet. If a boat is at \((3, 5)\), in what direction should the boat travel for the depth of the water to increase most rapidly and what would that rate of increase be?

**Solution:**

We have

\[
\nabla D(x, y) = -4x \mathbf{i} - 6y \mathbf{j}
\]

and so the direction would be

\[
\nabla D(3, 5) = -12 \mathbf{i} - 30 \mathbf{j}
\]

and the rate of increase would be

\[
||\nabla D(3, 5)|| = \sqrt{(-12)^2 + (-30)^2}
\]

(b) Ohm’s Law states that \( I = \frac{V}{R} \) which relates current \( (I) \) with voltage \( (V) \) and resistance \( (R) \). Suppose the voltage is decreasing at 5 volts/second while the resistance is decreasing at 2 ohms/second. Find the rate of change of the current with respect to time when the voltage is 80 volts and the resistance is 40 ohms.

**Solution:**

The chain rule tells us that

\[
\frac{dI}{dt} = \frac{\partial I}{\partial V} \frac{dV}{dt} + \frac{\partial I}{\partial R} \frac{dR}{dt}
\]

\[
= \frac{1}{R} (-5) - \frac{V}{R^2} (-2)
\]

\[
= \frac{1}{40} (-5) - \frac{80}{40^2} (-2)
\]
5. Let $R$ be the region in the $xy$-plane above the graph of $y = x^2$ and below the graph of $y = -(x - 1)^2 + 1$. Let $D$ be the solid above $R$ and below the plane $x + y + z = 5$.

(a) Separately sketch reasonable pictures of both $R$ and $D$. 

Solution:

(b) Set up an iterated double integral for the volume of $D$. Do not evaluate.

Solution:

If we parametrize $R$ as vertically simple then we get

$$
\int_0^1 \int_{x^2}^{-(x-1)^2+1} 5 - x - y \, dy \, dx
$$
6. Let $R$ be the parallelogram in the $xy$-plane formed by the lines $x + y = 1$, $x + y = 2$, $2y - x = 2$ and $2y - x = 0$.

(a) Sketch $R$.  

Solution:

(b) Use a change of variables to evaluate $\iint_{R} x + y \, dA$. Make sure to draw the new region in the $uv$-plane. This integral must be evaluated!

Solution:

We substitute $u = x + y$ and $v = 2y - x = -x + 2y$. This gives us the new region $S$ bounded by the lines $u = 1, 2$ and $v = 0, 2$:

Then we solve to get $x = \frac{2}{3}u - \frac{1}{3}v$ and $y = \frac{1}{3}u + \frac{1}{3}v$ so that

$$J = \begin{vmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{vmatrix} = 1/3$$

Alternately without solving for $x$ and $y$:

$$J = 1 \div \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 1/3$$

Then

$$\iint_{R} x + y \, dA = \iint_{S} u|1/3| \, dA$$

$$= \int_{1}^{2} \int_{0}^{2} \frac{1}{3}u \, dv \, du$$

$$= \int_{1}^{2} \frac{1}{3}uvw \bigg|_{0}^{2} \, du$$

$$= \int_{1}^{2} \frac{2}{3}u \, du = \frac{1}{3}u^2 \bigg|_{1}^{4} = \frac{1}{3}(4) - \frac{1}{3}(1)$$
7. Let \( C \) be the edge of the part of the plane \( 2x + 2y + z = 10 \) in the first octant, oriented counterclockwise when viewed from above.

(a) Apply Stokes’ Theorem to the integral \( \int_C 2y \, dx + x \, dy + xz \, dz \) to get a surface integral over a surface \( \Sigma \). Describe \( \Sigma \), including its induced orientation. Either words or a picture suffice.

**Solution:**

We have

\[
\int_C 2y \, dx + x \, dy + xz \, dz = \iint_{\Sigma} \left[ (0 - 0) \, i - (z - 0) \, j + (1 - 2) \, k \right] \cdot \bar{n} \, dS
\]

where \( \Sigma \) is the portion of the plane \( 2x + 2y + z = 10 \) in the first octant, oriented up and out.

(b) Parametrize \( \Sigma \) and convert your answer to (a) to an iterated double integral.

**Solution:**

Parametrize \( \Sigma \) as: \( \bar{r}(x, y) = x \, i + y \, j + (10 - 2x - 2y) \, k \) where \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 5 - x \). Then

\[
\bar{r}_x = 1 \, i + 0 \, j - 2 \, k \\
\bar{r}_y = 0 \, i + 1 \, j - 2 \, k \\
\bar{r}_x \times \bar{r}_y = 2 \, i + 2 \, j + 1 \, k
\]

these vectors match the orientation for \( \Sigma \) and so

\[
\iint_{\Sigma} \left[ (0 - 0) \, i - (z - 0) \, j + (1 - 2) \, k \right] \cdot \bar{n} \, dS = + \iint_R \left[ 0 \, i - (10 - 2x - 2y) \, j - 1 \, k \right] \cdot [2 \, i + 2 \, j + 1 \, k] \, dA
\]

\[
= \int_0^5 \int_0^{5-x} 4x + 4y - 21 \, dy \, dx
\]

(c) Evaluate.

**Solution:**

\[
\int_0^5 \int_0^{5-x} 4x + 4y - 21 \, dy \, dx = \int_0^5 4xy + 2y^2 - 21y \bigg|_0^{5-x} \, dx
\]

\[
= \int_0^5 4x(5 - x) + 2(5 - x)^2 - 21(5 - x) \, dx
\]

\[
= \int_0^5 20x - 4x^2 + 2x^2 - 20x + 50 - 105 + 21x \, dx
\]

\[
= \int_0^5 -2x^2 + 21x - 55 \, dx
\]

\[
= -\frac{2}{3}x^3 + 21x^2 - 55x \bigg|_0^5 = -\frac{2}{3}(5)^3 + \frac{21}{2}(5)^2 - 55(5)
\]
8. (a) Let $C$ be the part of the graph of the function $y = x^2$ from $x = 1$ to $x = 2$. Write down the [10 pts] iterated single integral corresponding to $\int_C x - y \, ds$. Do not evaluate.

**Solution:**
We parametrize as $\vec{r}(t) = t \, \mathbf{i} + t^2 \, \mathbf{j}$ for $1 \leq t \leq 2$.
Then $\vec{r}'(t) = 1 \, \mathbf{i} + 2t \, \mathbf{j}$ and $||\vec{r}'(t)|| = \sqrt{5}$.
Thus
$$\int_C x - y \, ds = \int_1^2 (t - t^2) \sqrt{5} \, dt$$

(b) Let $D$ be the solid inside the cone with spherical equation $\phi = \frac{\pi}{6}$ and below the plane $z = 3$. [15 pts] Let $\Sigma$ be the surface of $D$ oriented inwards. Apply the Divergence Theorem to the surface integral $\iint_{\Sigma} (x \, \mathbf{i} + xz \, \mathbf{j} + z^2 \, \mathbf{k}) \cdot \vec{n} \, dS$ and then use a spherical parametrization to obtain a triple iterated integral. Do not evaluate.

**Solution:**
By the Divergence Theorem (and due to orientation)
$$\iint_{\Sigma} (x \, \mathbf{i} + xz \, \mathbf{j} + z^2 \, \mathbf{k}) \cdot \vec{n} \, dS = - \iiint_D 1 + 0 + 2z \, dV$$
where $V$ is the solid inside the cone and below the plane.
Then the plane in spherical is $\rho = 3 / \cos \phi = 3 \sec \phi$ and so we get

$$= - \int_0^{2\pi} \int_0^{\pi/6} \int_0^{3\sec \phi} (2\rho \cos \phi + 1) \rho^2 \sin \phi \, d\rho d\phi d\theta$$