Math 241 Spring 2014 Final Exam Solutions

1. (a) Find the symmetric equation of the line containing \((1, 2, 3)\) and \((-1, 5, 3)\). [10 pts]

Solution:
\( \mathbf{L} = -2 \mathbf{i} + 3 \mathbf{j} + 0 \mathbf{k} \) so the symmetric equation is:
\[
\frac{x - 1}{-2} = \frac{y - 2}{3} , \quad z = 3
\]
Note: Other common answers may have a negated \( \mathbf{L} \) and may use the other point so please watch out for those!

(b) Find the distance between \((3, -5, 2)\) and the plane \(2x - y + 3z = 6\). Simplify. [10 pts]

Solution:
Normal vector for plane: \( \mathbf{N} = 2 \mathbf{i} - 1 \mathbf{j} + 3 \mathbf{k} \)
Point on plane: \( P = (3, 0, 0) \)
Point off plane: \( Q = (3, -5, 2) \)
We have
\[
\overrightarrow{PQ} = 0 \mathbf{i} - 5 \mathbf{j} + 2 \mathbf{k}
\]
and hence
\[
d = \frac{|\overrightarrow{PQ} \cdot \mathbf{N}|}{||\mathbf{N}||} = \frac{|0 + 5 + 6|}{\sqrt{4 + 1 + 9}} = \frac{11}{\sqrt{14}}
\]
Note: You’ll almost certainly see a variety of \( P \) and hence \( \overrightarrow{PQ} \) vectors. Probably you won’t see a different \( \mathbf{N} \).
2. (a) For \( \mathbf{u} = 2 \mathbf{i} - \mathbf{j} + 3 \mathbf{k} \) and \( \mathbf{v} = 4 \mathbf{i} + \mathbf{j} - 2 \mathbf{k} \), find \( P_{\mathbf{u}} \mathbf{v} \). [10 pts]

Solution:

\[
P_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{8 - 1 - 6}{4 + 1 + 9} (2 \mathbf{i} - \mathbf{j} + 3 \mathbf{k}) = \frac{1}{14} (2 \mathbf{i} - \mathbf{j} + 3 \mathbf{k})
\]

(b) Find the curvature \( \kappa(1) \) of \( \mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j} \). [10 pts]

Solution:

We have:

\[
\begin{align*}
\mathbf{r}(t) &= t^2 \mathbf{i} + t^3 \mathbf{j} \\
\mathbf{v}(t) &= 2t \mathbf{i} + 3t^2 \mathbf{j} \quad \text{and so} \quad \mathbf{v}(1) = 2 \mathbf{i} + 3 \mathbf{j} \\
\mathbf{a}(t) &= 2 \mathbf{i} + 6t \mathbf{j} \quad \text{and so} \quad \mathbf{a}(1) = 2 \mathbf{i} + 6 \mathbf{j}
\end{align*}
\]

and so

\[
\kappa(1) = \frac{||\mathbf{v}(1) \times \mathbf{a}(1)||}{||\mathbf{v}(1)||^3} = \frac{||6 \mathbf{k}||}{(\sqrt{4 + 9})^3} = \frac{6}{(13)^{3/2}}
\]
3. (a) Find $T(1)$ for $r(t) = t \mathbf{i} - 2t^3 \mathbf{j} + \frac{1}{t} \mathbf{k}$. \[5 \text{ pts}\]

**Solution:**

We have 

$$r'(t) = 1 \mathbf{i} - 6t^2 \mathbf{j} - \frac{1}{t^2} \mathbf{k}$$

and so 

$$T(1) = \frac{1 \mathbf{i} - 6 \mathbf{j} - 1 \mathbf{k}}{||1 \mathbf{i} - 6 \mathbf{j} - 1 \mathbf{k}||} = \frac{1 \mathbf{i} - 6 \mathbf{j} - 1 \mathbf{k}}{\sqrt{1 + 36 + 1}}$$

(b) Find the tangential component of acceleration for $r(t) = t^3 \mathbf{i} - 4t \mathbf{j} + t^2 \mathbf{k}$ at $t = 2$. \[5 \text{ pts}\]

**Solution:**

We have 

$$r(t) = t^3 \mathbf{i} - 4t \mathbf{j} + t^2 \mathbf{k}$$

$v(t) = 3t^2 \mathbf{i} - 4 \mathbf{j} + 2t \mathbf{k}$ and so $v(2) = 12 \mathbf{i} - 4 \mathbf{j} + 4 \mathbf{k}$

$a(t) = 6t \mathbf{j} + 0 \mathbf{j} + 2 \mathbf{k}$ and so $a(2) = 12 \mathbf{i} + 0 \mathbf{j} + 2 \mathbf{k}$

and so 

$$a_T(2) = \frac{v(2) \cdot a(2)}{||v(2)||} = \frac{144 + 0 + 8}{\sqrt{144 + 16 + 16}}$$

(c) Find the point at which the line $r(t) = (t + 1) \mathbf{i} - 2t \mathbf{j} + (3t - 2) \mathbf{k}$ passes through the plane $x + y - z = 10$. \[10 \text{ pts}\]

**Solution:**

The line hits the point when:

$$(t + 1) + (-2t) - (3t - 2) = 10$$

$$-4t = 7$$

$$t = -\frac{7}{4}$$

and this is at 

$$r(-\frac{7}{4}) = -\frac{3}{4} \mathbf{i} + \frac{7}{2} \mathbf{j} - \frac{29}{4} \mathbf{k}$$

Hence 

$$\left( -\frac{3}{4}, \frac{7}{2}, -\frac{29}{4} \right)$$
4. Use the method of Lagrange multipliers to find the maximum and minimum values of the function \( f(x, y) = xy \) on the circle \((x - 2)^2 + y^2 = 4\).

**Solution:**
Our system of equations is

\[
\begin{align*}
y &= \lambda(2) \quad (1) \\
x &= \lambda 2y \\
(x - 2)^2 + y^2 &= 4
\end{align*}
\]

To solve the first for \( \lambda \) we must divide by \( x - 2 \). If \( x - 2 = 0 \) then the first says \( y = 0 \) and \((2, 0)\) does not satisfy the third. Thus \( x - 2 \neq 0 \) and so the first yields \( \lambda = \frac{y}{2(x - 2)} \).

Plugging this into the second yields \( x = \frac{y}{2(x - 2)}(2y) = \frac{y^2}{x - 2} \) and so \( y^2 = x(x - 2) \).

Plugging this into the third yields \( (x - 2)^2 + x(x - 2) = 4 \) or \( 2x^2 - 6x = 0 \) or \( 2x(x - 3) = 0 \) or \( x = 0, 3 \).

Thus the points are \((0, 0)\) and \((3, \pm \sqrt{3})\).

Then

\[
\begin{align*}
f(0, 0) &= 0 \\
f(3, \sqrt{3}) &= 3\sqrt{3} \quad \text{The Maximum} \\
f(3, -\sqrt{3}) &= -3\sqrt{3} \quad \text{The Minimum}
\end{align*}
\]
5. Find and categorize all relative extrema for the function \( f(x, y) = x^3 - 2xy + y^2 \). \[20 \text{ pts}\]

**Solution:**

We have

\[
\begin{align*}
    f_x &= 3x^2 - 2y = 0 \\
    f_y &= -2x + 2y = 0 
\end{align*}
\]

The second yields \( y = x \) and so the first becomes \( x(3x - 2) = 0 \) and hence \( x = 0 \) or \( x = \frac{2}{3} \).

Thus the critical points are \((0, 0)\) and \((\frac{2}{3}, \frac{2}{3})\).

Then we have \( D(x, y) = (6x)(2) - (-2)^2 = 12x - 4 \) and so:

\( D(0, 0) = - \) so \((0, 0)\) is a saddle point.
\( D(\frac{2}{3}, \frac{2}{3}) = + \) and \( f_y(\frac{2}{3}, \frac{2}{3}) = + \) so \((\frac{2}{3}, \frac{2}{3})\) is a relative minimum.
6. Let \( f(x, y) = \ln(x^2 + xy + y^2) \).

(a) Find the direction of maximum increase of \( f \) at \((1, 0)\) as a unit vector. \([7 \text{ pts}]\)

**Solution:**
We have
\[
\nabla f(x, y) = \frac{2x + y}{x^2 + xy + y^2} \mathbf{i} + \frac{x + 2y}{x^2 + xy + y^2} \mathbf{j}
\]
\[
\nabla f(1, 0) = \frac{2}{1} \mathbf{i} + \frac{1}{1} \mathbf{j}
\]
\[
\nabla f(1, 0) = 2 \mathbf{i} + 1 \mathbf{j}
\]

and so the unit direction is:
\[
\frac{2 \mathbf{i} + 1 \mathbf{j}}{\sqrt{5}}
\]

(b) Find the maximum directional derivative at \((1, 0)\). \([6 \text{ pts}]\)

**Solution:**
We have
\[
||\nabla f(1, 0)|| = \sqrt{5}
\]

(c) Calculate the directional derivative of \( f \) at \((0, 1)\) in the direction of \(2 \mathbf{i} + 3 \mathbf{j}\). \([7 \text{ pts}]\)

**Solution:**
The appropriate unit vector is \(\frac{2 \mathbf{i} + 3 \mathbf{j}}{\sqrt{13}}\) and \(\nabla f(0, 1) = 1 \mathbf{i} + 2 \mathbf{j}\) and so
\[
D_u f(0, 1) = \left( \frac{2}{\sqrt{13}} \mathbf{i} + \frac{3}{\sqrt{13}} \mathbf{j} \right) \cdot (1 \mathbf{i} + 2 \mathbf{j}) = \frac{8}{\sqrt{13}}
\]
7. (a) Find a parametrization for the part of the cylinder $y^2 + z^2 = 1$ which lies between $x = -2$ and $x = 2$. [5 pts]

Solution:
Perhaps the most obvious possibility is:

$$\mathbf{r}(x, \theta) = x \mathbf{i} + \cos \theta \mathbf{j} + \sin \theta \mathbf{k}$$

$$-2 \leq x \leq 2$$
$$0 \leq \theta \leq 2\pi$$

(b) Find the equation of the plane tangent to the cylinder in part (a) at the point $\left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. [15 pts]

Write your answer in the form $ax + by + cz = d$.

Solution:
The cylinder is the level surface for $f(x, y, z) = y^2 + z^2 = 1$ and hence the normal vector is:

$$\nabla f(x, y, z) = 0 \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k}$$

$$\nabla f \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0 \mathbf{i} + \sqrt{2} \mathbf{j} + \sqrt{2} \mathbf{k}$$

and so the plane is

$$0(x - 1) + \sqrt{2} \left(y - \frac{1}{\sqrt{2}}\right) + \sqrt{2} \left(z - \frac{1}{\sqrt{2}}\right) = 0$$

$$\sqrt{2}y + \sqrt{2}z = 2$$

Note: A student may write $y + z = \sqrt{2}$ instead, or other variations.
8. Find the volume of the solid region $D$ that is bounded on the sides by the upper nappe of the cone $z^2 = \frac{1}{3}(x^2 + y^2)$, on the top by the sphere $x^2 + y^2 + z^2 = 9$ and below by the sphere $x^2 + y^2 + z^2 = 1$.

Solution:
We have:

$$Volume = \iiint_D 1 \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \rho^3 \sin \phi \bigg|_0^1 \, d\phi \, d\theta$$

$$= \frac{26}{3} \int_0^{2\pi} \sin \phi \bigg|_0^{\pi/3} \, d\theta$$

$$= \frac{26}{3} \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \frac{13}{3} \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{26\pi}{3}$$
9. Let \( C \) be the intersection curve of the parabolic sheet \( y = x^2 \) with the cylinder \( x^2 + z^2 = 4 \), oriented clockwise when viewed from the positive \( y \)-axis. Apply Stokes’ Theorem to the integral \( \int_C 2y \, dx + xz \, dy + z^2 \, dz \) and continue until you have an iterated double integral. Do not evaluate.

**Solution:**

Stokes’ Theorem gives us:

\[
\int_{C} 2y \, dx + xz \, dy + z^2 \, dz = \int_{\Sigma} (-x \, i + 0 \, j + (z - 2) \, k) \cdot n \, dS
\]

Where \( \Sigma \) is the portion of the parabolic sheet inside the cylinder, oriented to the left. We parametrize \( \Sigma \) as

\[
\mathbf{r}(r, \theta) = r \cos \theta \, i + r^2 \cos^2 \theta \, j + r \sin \theta \, k \quad \text{with} \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi
\]

And so

\[
\mathbf{r}_r = \cos \theta \, i + 2r \cos^2 \theta \, j + \sin \theta \, k \\
\mathbf{r}_\theta = -r \sin \theta \, i - 2r^2 \sin \theta \cos \theta \, j + r \cos \theta \, k \\
\mathbf{r}_r \times \mathbf{r}_\theta = 2r^2 \cos \theta \, i - r \, j + 0 \, k
\]

This matches the orientation of \( \Sigma \) and hence

\[
\int_{\Sigma} (-x \, i + 0 \, j + (z - 2) \, k) \cdot n \, dS = \int_{R} (-r \cos \theta \, i + 0 \, j + (r \sin \theta - 2) \, k) \cdot (2r^2 \cos \theta \, i - r \, j + 0 \, k) \, dA = \int_{0}^{2\pi} \int_{0}^{2} (-r \cos \theta)(2r^2 \cos \theta) + (0)(-r) + (0)(r \sin \theta - 2)(0) \, dr \, d\theta
\]
10. (a) Evaluate \( \int_C 7y \, dx + 12y \, dy \) where \( C \) is the semicircle \( y = \sqrt{9-x^2} \) along with the line segment joining \((-3,0)\) with \((3,0)\), oriented clockwise.

**Solution:**
We apply Green’s Theorem with a negative sign due to the orientation:
\[
\int_C 7y \, dx + 12y \, dy = - \iint_R 0 - 7 \, dA = 7(Area \ of \ R) = 7 \left( \frac{1}{2} \pi 3^2 \right)
\]

(b) Find the surface area of the portion of the sphere \( x^2 + y^2 + z^2 = 4 \) inside the cylinder \( x^2 + y^2 - 2y = 0 \) as an iterated double integral in \( r \) and \( \theta \). Do not evaluate.

**Solution:**
The cylinder is \( x^2 + (y - 2)^2 = 4 \) or \( r = 2 \sin \theta \). We therefore parametrize the top part of the surface (which we’ll double) as
\[
r(r, \theta) = r \cos \theta \, \mathbf{i} + r \sin \theta \, \mathbf{j} + \sqrt{4-r^2} \, \mathbf{k} \\
0 \leq \theta \leq \pi \\
0 \leq r \leq 2 \sin \theta
\]
and so
\[
\mathbf{r}_r = \cos \theta \, \mathbf{i} + \sin \theta \, \mathbf{j} + 2r(4-r^2)^{-1/2} \, \mathbf{k} \\
\mathbf{r}_\theta = -r \sin \theta \, \mathbf{i} + r \cos \theta \, \mathbf{j} + 0 \, \mathbf{k} \\
\mathbf{r}_r \times \mathbf{r}_\theta = 2r^2 \cos \theta(4-r^2)^{-1/2} \, \mathbf{i} + 2r^2 \sin \theta(4-r^2)^{-1/2} \, \mathbf{j} + r \, \mathbf{k}
\]
\[
|| \mathbf{r}_r \times \mathbf{r}_\theta || = \sqrt{4r^4 \cos^2 \theta(4-r^2)^{-1} + 4r^4 \sin^2 \theta(4-r^2)^{-1} + r^2}
\]
\[
= \sqrt{4r^2(4-r^2)^{-1} + r^2}
\]
And so
\[
SA = 2 \iint_{\Sigma} 1 \, dS = 2 \iint_R \sqrt{4r^2(4-r^2)^{-1} + r^2} \, dA = 2 \int_0^\pi \int_0^{2\sin \theta} \sqrt{4r^2(4-r^2)^{-1} + r^2} \, dr \, d\theta
\]

Note: This could be done with the shortcut formula from the book and then converted to polar. This would probably end up with the \( r^2 \) pulled to the outside as \( r \) since it arises as the Jacobian.