1. (a) Find the point where the line with parametric equations $x = 1 + 2t$, $y = 2 - t$, $z = 4 - 2t$ meets the plane $x + y - 2z = 10$. [10 pts]

Solution: We substitute and solve:

$$\begin{aligned} (1 + 2t) + (2 - t) - 2(4 - 2t) &= 10 \\ -5 + 5t &= 10 \\ 5t &= 15 \\ t &= 3 \end{aligned}$$

So the point is at $x = 1 + 2(3) = 7$, $y = 2 - 3 = -1$ and $z = 4 - 2(3) = -2$, that is $(7, -1, -2)$.

(b) Find the symmetric equations of the line perpendicular to the plane $x + y - 2z = 10$ and passing through the point $(1, 2, 3)$. [10 pts]

Solution: The direction vector for the line can be the same as the normal vector for the plane:

$$\mathbf{L} = 1\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k}$$

Thus the parametric equations (if they write them first) would be

$$\begin{aligned} x &= 1t + 1 \\ y &= 1t + 2 \\ z &= -2t + 3 \end{aligned}$$

and the symmetric equations would be

$$\frac{x - 1}{1} = \frac{y - 2}{1} = \frac{z - 3}{-2}$$

or

$$x - 1 = y - 2 = \frac{3 - z}{2}$$

2. (a) Find an equation of the plane passing through the points $(-2, 1, 1)$, $(0, 2, 3)$ and $(1, 0, -1)$. [10 pts]

Solution: If we call the points P, Q and R respectively and then we construct two vectors and take the cross product to find $\mathbf{N}$:

$$\begin{aligned}
\vec{PQ} &= 2\,\mathbf{i} + 1\,\mathbf{j} + 2\,\mathbf{k} \\
\vec{PR} &= 3\,\mathbf{i} - 1\,\mathbf{j} - 2\,\mathbf{k} \\
\mathbf{N} &= (-2 + 2)\,\mathbf{i} - (-4 - 6)\,\mathbf{j} + (-2 - 3)\,\mathbf{k} \\
&= 0\,\mathbf{i} + 10\,\mathbf{j} - 5\,\mathbf{k}
\end{aligned}$$

Then using P the plane would have equation (simplification not necessary):

$$\begin{aligned}
0(x - (-2)) + 10(y - 1) - 5(z - 1) &= 0 \\
10(y - 1) - 5(z - 1) &= 0 \\
10y - 5z &= 5 \\
2y - z &= 1
\end{aligned}$$

(b) Let $\mathbf{a} = 2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k}$ and $\mathbf{b} = 0\,\mathbf{i} + 6\,\mathbf{j} + 10\,\mathbf{k}$. Find $\mathbf{pr_a b}$. [10 pts]

Solution: We have:

$$\begin{aligned}
\mathbf{pr_a b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} \\
&= \frac{(2)(0) + (3)(6) + (-1)(10)}{2^2 + 3^2 + (-1)^2}(2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k}) \\
&= \frac{8}{14}(2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k}) \\
&= \frac{8}{7}\,\mathbf{i} + \frac{12}{7}\,\mathbf{j} - \frac{4}{7}\,\mathbf{k}
\end{aligned}$$

3. Suppose C is parametrized by $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$ for $0 \le t \le 3$.

(a) Find the length of the curve C. Simplify. [10 pts]

Solution: To find the length we first find:

$$\begin{aligned}
\mathbf{r}'(t) &= (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j} \\
||\mathbf{r}'(t)|| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \\
&= \sqrt{e^{2t}\cos^2 t - 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t} \\
&= \sqrt{2e^{2t}} \\
&= \sqrt{2}e^t
\end{aligned}$$

and so

$$\begin{aligned}
\text{Length} &= \int_0^3 \sqrt{2}e^t \; dt \\
&= \sqrt{2}e^t \Big|_0^3 \\
&= \sqrt{2}e^3 - \sqrt{2}e^0
\end{aligned}$$

(b) Find the unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{N}(t)$. Simplify. [10 pts]

Solution: From above we have

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \\
&= \frac{1}{\sqrt{2}}\left[(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}\right]
\end{aligned}$$

and from there:

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{1}{\sqrt{2}}\left[(-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\right] \\
||\mathbf{T}'(t)|| &= \sqrt{\frac{1}{2}(-\sin t - \cos t)^2 + \frac{1}{2}(\cos t - \sin t)^2} \\
&= \sqrt{\frac{1}{2}\left[\sin^2 t + 2\sin t\cos t + \cos^2 t + \cos^2 t - 2\sin t\cos t + \sin^2 t\right]} = 1
\end{aligned}$$

and so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{1}{\sqrt{2}}\left[(-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\right]$$

4. Use the Fundamental Theorem of Line Integrals to evaluate $\int_C (2xy+z)\ dx + x^2\ dy + x\ dz$ where C is parametrized by $\mathbf{r}(t) = 16t^2\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + (2t-1)\,\mathbf{k}$ for $\frac{1}{2} \leq t \leq 1$. [20 pts]

Solution: The potential function is given by

$$f(x, y, z) = x^2y + xz$$

The endpoints of the curve are given by

$$\text{Start: } \mathbf{r}\left(\frac{1}{2}\right) = 4\,\mathbf{i} + 2\,\mathbf{j} + 0\,\mathbf{k} \qquad (4, 2, 0)$$

$$\text{End: } \mathbf{r}(1) = 16\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k} \qquad (16, 1, 1)$$

And so

$$\begin{aligned}\int_C (2xy + z)\ dx + x^2\ dy + x\ dz &= f(16, 1, 1) - f(4, 2, 0) \\ &= \left[16^2(1) + 16(1)\right] - \left[4^2(2) + 4(0)\right] \\ &= 272 - 32 \\ &= 240\end{aligned}$$

5. The object distance $x > 0$, image distance $y > 0$ and focal length L of a simple lens satisfy: [20 pts]

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{L}$$

Using Lagrange multipliers find the minimum of $f(x, y) = x + y$ subject to the constraint above. You may assume that the minimum exists and that L is a fixed constant.

Solution: The constraint function is given by

$$g(x, y) = \frac{1}{x} + \frac{1}{y} - \frac{1}{L}$$

So we solve the system

$$1 = \lambda\left(-\frac{1}{x^2}\right) \tag{1}$$

$$1 = \lambda\left(-\frac{1}{y^2}\right) \tag{2}$$

$$0 = \frac{1}{x} + \frac{1}{y} - \frac{1}{L} \tag{3}$$

Equation (1) tells us that $\lambda = -x^2$ and equation (2) tells us that $\lambda = -y^2$. Therefore $x^2 = y^2$ and since they're both positive, $x = y$.

Then equation (3) tells us that $\frac{1}{x} + \frac{1}{x} = \frac{1}{L}$ so that $x = \frac{1}{2L}$ and so $y = \frac{1}{2L}$ also.

The minimum is then $f\left(\frac{1}{2L}, \frac{1}{2L}\right) = \frac{1}{L}$.

6. (a) Let $z(x,y) = x^2 - xy^2$. For all (x,y) compute [10pts]

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

Solution: We have

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x - y^2\\ \frac{\partial z}{\partial y} &= -2xy\\ \frac{\partial^2 z}{\partial x^2} &= 2\\ \frac{\partial^2 z}{\partial y^2} &= -2x\\ \frac{\partial^2 z}{\partial x \partial y} &= -2y\end{aligned}$$

so that

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (2)(-2x) - (-2y)^2$$

(b) By differentiating both sides of the equation [10pts]

$$f(tx,ty) = t^2 f(x,y)$$

with respect to t and then setting $t = 1$, show that

$$x\frac{\partial f(x,y)}{\partial x} + y\frac{\partial f(x,y)}{\partial y} = 2f(x,y)$$

Solution: By the chain rule we have

$$\begin{aligned}f(tx,ty) &= t^2 f(x,y)\\ \frac{d}{dt}f(tx,ty) &= \frac{d}{dt}t^2 f(x,y)\\ \frac{\partial f}{\partial x}(tx,ty)(x) + \frac{\partial f}{\partial y}(tx,ty)y &= 2tf(x,y)\end{aligned}$$

Then when $t = 1$ we get

$$\frac{\partial f}{\partial x}(x,y)(x) + \frac{\partial f}{\partial y}(x,y)y = f(x,y)$$

7. Let R be the region in the $xy-$plane between the graphs of $y = x^2$ and $y = 1 - x^2$. Let D be the solid region between R and the parabolic sheet $z = x^2$. Find the volume of D. Simplify as much as possible. [20 pts]

Solution: The region R is bounded on the left and right by $\pm\sqrt{\frac{1}{2}}$ found by solving $x^2 = 1 - x^2$.

The volume is therefore given by:

$$\begin{aligned}
\text{Volume} &= \iiint_D 1 \; dV \\
&= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \int_{x^2}^{1-x^2} \int_0^{x^2} 1 \; dz \; dy \; dx \\
&= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} \int_{x^2}^{1-x^2} x^2 \; dy \; dx \\
&= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} x^2 y \Big|_{x^2}^{1-x^2} x^2 \; dx \\
&= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} x^2(1 - x^2) - x^2(x^2) \; dx \\
&= \int_{-\sqrt{1/2}}^{\sqrt{1/2}} x^2 - 2x^4 \; dx \\
&= \frac{1}{3}x^3 - \frac{2}{5}x^5 \Big|_{-\sqrt{1/2}}^{\sqrt{1/2}} \\
&= \left[\frac{1}{3}\left(\sqrt{1/2}\right)^3 - \frac{2}{5}\left(\sqrt{1/2}\right)^5\right] - \left[\frac{1}{3}\left(-\sqrt{1/2}\right)^3 - \frac{2}{5}\left(-\sqrt{1/2}\right)^5\right]
\end{aligned}$$

8. Let D be the solid region inside the sphere $\rho = 2$ and inside the cone $z = \sqrt{x^2 + y^2}$. Evaluate the integral $\iiint_D z^2 \ dV$ using spherical coordinates. [20 pts]

Solution: We have:

$$\begin{aligned}
\iiint\limits_D z^2 \ dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi)^2 \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{5}\rho^5 \sin \phi \cos^2 \phi \Big|_0^2 \ d\phi \ d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{32}{5} \sin \phi \cos^2 \phi \ d\phi \ d\theta \\
&= \int_0^{2\pi} -\frac{32}{15} \cos^3 \phi \Big|_0^{\pi/4} \ d\theta \\
&= \int_0^{2\pi} -\frac{32}{15} \left[\cos^3(\pi/4) - \cos^3(0)\right] \ d\theta \\
&= \int_0^{2\pi} -\frac{32}{15} \left[\frac{\sqrt{2}}{4} - 1\right] \ d\theta \\
&= -\frac{32}{15} \left[\frac{\sqrt{2}}{4} - 1\right] \theta \Big|_0^{2\pi} \\
&= -\frac{32}{15} \left[\frac{\sqrt{2}}{4} - 1\right] (2\pi)
\end{aligned}$$

9. Let C be the intersection of the cylinder $x^2 + z^2 = 4$ with the plane $x + y = 4$ and with counterclockwise orientation when viewed from the positive y-axis. Use Stokes' Theorem to convert the line integral [20 pts]

$$\int_C xy\ dx + y\ dy + xz\ dz$$

to a surface integral. Write your integral as an iterated integral in polar coordinates. Do not evaluate this integral.

Solution: By Stokes' Theorem:

$$\int_C xy\ dx + y\ dy + xz\ dz = \iint_\Sigma (0\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot \mathbf{n}\ dS$$

where Σ is the part of the plane inside the cylinder with orientation forward and to the right.

We parametrize Σ by

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + (4 - r\cos\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k}$$
$$0 \le \theta \le 2\pi,\ 0 \le r \le 2 \text{ (This is our } R\text{)}$$

So then

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\,\mathbf{i} - \cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k}\\ \mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\cos\theta\,\mathbf{k}\\ \mathbf{r}_r \times \mathbf{r}_\theta &= -r\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k}\end{aligned}$$

which is opposite of Σ's orientation, thus:

$$\begin{aligned}\iint_\Sigma (0\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot \mathbf{n}\ dS &= -\iint_R (0\,\mathbf{i} - (r\sin\theta)\,\mathbf{j} - (r\cos\theta)\,\mathbf{k}) \cdot (-r\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k})\ dA\\ &= -\int_0^{2\pi}\int_0^2 r^2\sin\theta\ dr\ d\theta\end{aligned}$$

10. (a) Let C be the triangle in the xy-plane with corners $(0,0)$, $(4,2)$ and $(0,6)$, oriented clockwise. [10 pts] By using Green's Theorem calculate the line integral $\int_C 3xy\ dx + 4x^2\ dy$.

Solution: If R is the filled-in triangle inside C then, noting the orientation of C, by Green's Theorem:

$$\begin{aligned}\int_C 3xy\ dx + 4x^2\ dy &= -\iint_R 8x - 3x\ dA \\ &= -\iint_R 5x\ dA \\ &= -\int_0^4 \int_{\frac{1}{2}x}^{6-x} 5x\ dy\ dx \\ &= -\int_0^4 5xy\Big|_{\frac{1}{2}x}^{6-x}\ dx \\ &= -\int_0^4 5x(6-x) - 5x\left(\frac{1}{2}x\right)\ dx \\ &= -\int_0^4 -\frac{15}{2}x^2 + 30x\ dx \\ &= \frac{5}{2}x^3 - 15x^2\Big|_0^4 \\ &= \frac{3}{2}(4)^3 - 15(4)^2\end{aligned}$$

(b) Evaluate $\int_C x^2y\ ds$ where C is the circle $x^2 + y^2 = 4$. [10 pts]

Solution: We parametrize the curve by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\mathbf{r}'(t) &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \\ ||\mathbf{r}'(t)|| &= 2\end{aligned}$$

and so

$$\begin{aligned}\int_C x^2y\ ds &= \int_0^{2\pi} (2\cos t)^2(2\sin t)(2)\ dt \\ &= \int_0^{2\pi} 16\cos^2 t\sin t\ dt \\ &= -\frac{16}{3}\cos^3 t\Big|_0^{2\pi} \\ &= 0\end{aligned}$$