Math 241 Fall 2017 Final Exam Solutions

1. (a) Find the cosine of the angle between the vectors $\mathbf{a} = 2\,\mathbf{i} - 1\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{b} = 0\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$. [5 pts]
Simplify your answer as much as possible.
Solution:

$$\begin{aligned}\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}\\ &= \frac{(2)(0)+(-1)(4)+(3)(2)}{\sqrt{2^2+(-1)^2+3^2}\sqrt{0^2+4^2+2^2}}\\ &= \frac{2}{\sqrt{14}\sqrt{20}}\\ &= \frac{1}{\sqrt{70}}\end{aligned}$$

(b) Find the distance from the point $(3, 1, -1)$ to the plane with equation $2x - y - 2z = 4$. [20 pts]
Simplify your answer as much as possible.
Solution:
We set:

$$Q = (3, 1, -1) \text{ off the plane}$$

and

$$P = (2, 0, 0) \text{ on the plane}$$

Then we have

$$\overline{PQ} = 1\,\mathbf{i} + 1\,\mathbf{j} - 1\,\mathbf{k}$$

and

$$\mathbf{N} = 2\,\mathbf{i} - 1\,\mathbf{j} - 2\,\mathbf{k}$$

so that

$$\text{dist} = \frac{\left|\overline{PQ}\cdot\mathbf{N}\right|}{||\mathbf{N}||} = \frac{3}{\sqrt{(2)^2+(-1)^2+(-2)^2}} = 1$$

2. Find the equation of the line of intersection between the planes $x + 4z = 1$ and $-x + 2y + 3z = 5$. [20 pts]
Solution:
The direction vector for the line can be found by crossing the two planes' normal vectors:

$$\begin{aligned}\mathbf{L} &= (1\,\mathbf{i} + 0\,\mathbf{j} + 4\,\mathbf{k}) \times (-1\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k})\\ &= -8\,\mathbf{i} - 7\,\mathbf{j} + 2\,\mathbf{k}\end{aligned}$$

Since $x = 1$, $z = 0$ satisfies the first plane we can find $y = 3$ for the second plane.

The parametric equations are then:

$$\begin{aligned} x &= 1 - 8t \\ y &= 3 - 7t \\ z &= 0 + 2t \end{aligned}$$

In vector form:

$$\mathbf{r}(t) = (1 - 8t)\,\mathbf{i} + (3 - 7t)\,\mathbf{j} + (0 + 2t)\,\mathbf{k}$$

In symmetric form this would be:

$$\frac{x-1}{-8} = \frac{y-3}{-7} = \frac{z}{2}$$

3. The position of a particle at time $t \geq 0$ is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k}$.

   (a) Find the length of the curve traced out by the particle in the time interval $0 \leq t \leq 3$. [10 pts]
   Simplify your answer as much as possible.
   Solution:
   We have

$$r'(t) = 2\,\mathbf{i} + 2t\,\mathbf{j} - t^2\,\mathbf{k}$$

$$\begin{aligned} \mathit{length} &= \int_0^3 ||2\,\mathbf{i} + 2t\,\mathbf{j} - t^2\,\mathbf{k}||\,dt \\ &= \int_0^3 \sqrt{2 + 4t^2 + t^4}\,dt \\ &= \int_0^3 t^2 + 2\,dt \\ &= \frac{1}{3}t^3 + 2t\Big|_0^3 \\ &= \frac{1}{3}(27) + 2(3) \\ &= 15 \end{aligned}$$

   (b) Find the the tangential component of acceleration. [5 pts]
   Simplify your answer as much as possible.
   Solution:

$$\begin{aligned} a_{\mathbf{T}} &= \frac{\mathbf{v} \cdot \mathbf{a}}{||\mathbf{v}||} \\ &= \frac{(2\,\mathbf{i} + 2t\,\mathbf{j} - t^2\,\mathbf{k}) \cdot (0\,\mathbf{i} + 2\,\mathbf{j} - 2t\,\mathbf{k})}{t^2 + 2} \\ &= \frac{4t + 2t^3}{t^2 + 2} \\ &= 2t \end{aligned}$$

(c) Is the speed of the particle increasing or decreasing when $t = 3$? Explain. [5 pts]

Solution:

Since speed equals $s(t) = t^2 + 2$ we have $s'(t) = 2t$ and $s(3) = 6 > 0$ so the particle is speeding up.

4. Parts (a) and (b) are independent problems.

(a) Let $w = f(x, y)$ where $x = e^{-s} \cos t$ and $y = e^{-s} \sin t$, and $f(x, y)$ is a function with continuous partial derivatives of first and second order. Compute both the partial derivatives w_s and w_t in terms of x, y, f_x and f_y. [10 pts]

Solution:

By the chain rule we have

$$\begin{aligned} w_s &= f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} \\ &= f_x(-e^{-s} \cos t) + f_y(-e^{-s} \sin t) \\ &= -x f_x - y f_y \end{aligned}$$

$$\begin{aligned} w_t &= f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} \\ &= f_x(-e^{-s} \sin t) + f_y(e^{-s} \cos t) \\ &= -y f_x + x f_y \end{aligned}$$

(b) Find a unit vector perpendicular to the graph of $f(x, y) = xy + xy^2$ at the point where $x = 1$ and $y = -2$. [10 pts]

Simplify your answer as much as possible.

Solution:

The graph of f is a level surface for

$$g(x, y, z) = xy + xy^2 - z$$

so then

$$\begin{aligned} \nabla g &= (y + y^2)\,\mathbf{i} + (x + 2xy)\,\mathbf{j} - 1\,\mathbf{k} \\ \nabla g(1, -2, *) &= 2\,\mathbf{i} - 2\,\mathbf{j} - 1\,\mathbf{k} \end{aligned}$$

and so a unit vector is

$$\frac{\nabla g(1,-2,*)}{||\nabla g(1,-2,*)||} = \frac{2\,\mathbf{i} - 2\,\mathbf{j} - 1\,\mathbf{k}}{||2\,\mathbf{i} - 2\,\mathbf{j} - 1\,\mathbf{k}||} = \frac{2}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} - \frac{1}{3}\,\mathbf{k}$$

5. Consider the function $f(x,y) = 3x^2 + 2y^2 - 8y + 1$. Find the maximum and minimum values of f on the region R of the xy-plane described by $x^2 + y^2 \leq 1$. [20 pts]

   Solution:

   First we have

$$\begin{aligned} f_x &= 6x \\ f_y &= 4y - 8 \end{aligned}$$

   so the only critical point is at $(0,2)$ which is not in the domain.

   On the edge we have $x^2 = 1 - y^2$ so that

$$f(y) = 3(1-y^2) + 2y^2 - 8y + 1 = -y^2 - 8y + 4 \text{ for } -1 \leq y \leq 1$$

   Then $f'(y) = -2y - 8$ which equals 0 at $y = 4$, outside the domain, so we check

$$\begin{aligned} f(-1) &= 13 \\ f(1) &= -3 \end{aligned}$$

   So the maximum is 13 and the minimum is -3.

6. Parts (a) and (b) are independent problems.

   (a) Set up a triple iterated integral that would compute the volume of the solid region under the paraboloid $z = 9 - x^2 - y^2$, above the plane $z = 1$, and for $y \geq 0$. [10 pts]

   Do not evaluate this integral.

   Solution:

   The paraboloid meets the plane when

$$\begin{aligned} 9 - x^2 - y^2 &= 1 \\ x^2 + y^2 &= 8 \end{aligned}$$

   and so we have

$$volume = \int_0^{\pi} \int_0^{\sqrt{8}} \int_1^{9-r^2} r\,dz\,dr\,d\theta$$

(b) Set up a triple iterated integral that would determine the volume of the solid region below $z = x^4 + y^4 + 4$ and above the triangular region on the xy-plane with vertices $(-2, 0)$, $(2, 0)$, and $(0, 2)$. [10 pts]

Do not evaluate this integral.

Solution:

Treating the triangular region as horizontally simple we have

$$volume = \int_0^2 \int_{y-2}^{2-y} \int_0^{x^2+y^2+4} 1\, dz\, dx\, dy$$

7. Use an appropriate change of variables to evaluate [25 pts]

$$\iint_R (3x + 6y)^2\, dA$$

where R is the parallelogram enclosed by the lines $x - 2y = -2$, $x - 2y = 2$, $\frac{1}{2}x + y = 1$, and $\frac{1}{2}x + y = -1$.

Evaluate and simplify your answer as much as possible.

Solution:

If we set $u = x - 2y$ and $v = \frac{1}{2}x + y$ then our lines under the transformation are

$$\begin{aligned} x - 2y = -2 &\to u = -2 \\ x - 2y = 2 &\to u = 2 \\ \frac{1}{2}x + y = 1 &\to v = 1 \\ \frac{1}{2}x + y = -1 &\to v = -1 \end{aligned}$$

The integrand is then $(3x + 6y)^2 = (6v)^2 = 36v^2$ and the Jacobian is:

$$1 \div det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = 1 \div det \begin{bmatrix} 1 & -2 \\ 1/2 & 1 \end{bmatrix} 1 \div 2 = \frac{1}{2}$$

and so we have

$$\begin{aligned}
\iint_R (3x+6y)^2\,dA &= \iint_S 36v^2 \left|\frac{1}{2}\right|\,dA \\
&= \int_{-2}^{2}\int_{-1}^{1} 18v^2\,dv\,du \\
&= \int_{-2}^{2} 6v^3\Big|_{-1}^{1}\,du \\
&= \int_{-2}^{2} 12\,du \\
&= 12u\Big|_{-2}^{2} \\
&= 48
\end{aligned}$$

8. Parts (a) and (b) are independent problems.

   (a) Let C be the clockwise curve consisting of the quarter-circle $x^2+y^2=4$ with $x, y \geq 0$ along with the line segment joining $(0,0)$ to $(0,2)$ and the line segment joining $(0,0)$ to $(2,0)$. Use Green's Theorem to evaluate $\int_C 2x\,dx + x^2\,dy$. [15 pts]
   Evaluate and simplify your answer as much as possible.
   Solution:
   By Green's Theorem if R is the quarter-disk then we have, including $-$ for the orientation switch,

$$\begin{aligned}
\int_C 2x\,dx + x^2\,dy &= -\iint_R 2x\,dA \\
&= -\int_0^{\pi/2}\int_0^2 2r^2\cos\theta\,dr\,d\theta \\
&= -\int_0^{\pi/2} \frac{2}{3}r^3\cos\theta\Big|_0^2\,d\theta \\
&= -\int_0^{\pi/2} \frac{16}{3}\cos\theta\,d\theta \\
&= -\frac{16}{3}\sin\theta\Big|_0^{\pi/2} = -\frac{16}{3}
\end{aligned}$$

   (b) Let Σ be the part of the plane $2x+y+3z=12$ in the first octant. If the mass density at any point is given by $f(x,y,z)=xz$, set up a double iterated integral for the mass of Σ. [10 pts]
   Do not evaluate this integral.
   Solution:
   The surface can be parametrized by

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \left(4 - \tfrac{2}{3}x - \tfrac{1}{3}y\right)\,\mathbf{j} \text{ with } 0 \leq x \leq 6 \text{ and } 0 \leq y \leq 12-2x \text{ (this is } R\text{)}$$

Therefore we have

$$\begin{aligned}
\mathbf{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} - \frac{2}{3}\,\mathbf{k} \\
\mathbf{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} - \frac{1}{3}\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= \frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} + 1\,\mathbf{j} \\
||\mathbf{r}_x \times \mathbf{r}_y|| &= \sqrt{\frac{2}{9} + \frac{1}{9} + 1}
\end{aligned}$$

and so

$$mass = \iint_\Sigma xz\,dS = \iint_R x\left(4 - \frac{2}{3}x - \frac{1}{3}y\right)\,dA = \int_0^6 \int_0^{12-2x} x\left(4 - \frac{2}{3}x - \frac{1}{3}y\right)\,dy\,dx$$

9. Parts (a) and (b) are independent problems.

(a) Let C be the intersection of the cylinder $x^2 + z^2 = 4$ with the plane $y + z = 8$ and with counterclockwise orientation when viewed from above. Use Stokes' Theorem to convert the line integral [20 pts]

$$\int_C x^2 z\,dx + x\,dy + yz\,dz$$

to a surface integral. Parametrize the surface to obtain a double iterated integral.
Do not evaluate this integral.
Solution:
By Stokes' Theorem we have

$$\int_C x^2 z\,dx + x\,dy + yz\,dz = \iint_\Sigma (z\,\mathbf{i} + x^2\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n}\,dS$$

where Σ is the part of the plane $y + z = 8$ inside the cylinder with induced orientation.

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + (8 - r\sin\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k} \text{ with } 0 \le \theta \le 2\pi \text{ and } 0 \le r \le 2 \text{ (this is } R)$$

Therefore we have

$$\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} + \sin\theta\,\mathbf{k} \\
\mathbf{r}_y &= -r\sin\theta\,\mathbf{i} - r\cos\theta\,\mathbf{j} + r\cos\theta\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= 0\,\mathbf{i} - r\,\mathbf{j} - r\,\mathbf{k}
\end{aligned}$$

This orientation is opposite of the induced orientation and so

$$\begin{aligned}
\iint_\Sigma (z\,\mathbf{i} + x^2\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n}\,dS &= -\iint_R \left[r\sin\theta\,\mathbf{i} + r^2\cos^2\theta\,\mathbf{j} + 1\,\mathbf{k}\right] \cdot \left[0\,\mathbf{i} - r\,\mathbf{j} - r\,\mathbf{k}\right]\,dA \\
&= -\int_0^{2\pi} \int_0^2 -r^3\cos^2\theta - r\,dr\,d\theta
\end{aligned}$$

(b) Evaluate $\iint_\Sigma (2x\,\mathbf{i} + 3x\,\mathbf{j} - 5z\,\mathbf{k}) \cdot \mathbf{n}\,dS$ where Σ is the sphere $x^2 + y^2 + z^2 = 9$ with $\mathbf{n}$ the unit normal vector pointing inwards. That is, the sphere is oriented inwards. [5 pts]

Evaluate and simplify your answer as much as possible.

Solution:

By the Divergence Theorem and considering the orientation of Σ treated as the surface (boundary) surrounding D, the sphere of radius 3 centered at the origin, we have

$$\iint_\Sigma (2x\,\mathbf{i} + 3x\,\mathbf{j} - 5z\,\mathbf{k}) \cdot \mathbf{n}\,dS = -\iiint_D 2 + 0 - 5\,dV = 3\,vol(D) = 3\left(\frac{4}{3}\pi(3)^3\right) = 108\pi$$