Math 241 Spring 2017 Final Exam Solution

1. (a) Find $\cos\theta$ where θ is the angle between the vectors $\mathbf{a} = 2\,\mathbf{i} + 1\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = 1\,\mathbf{i} + 5\,\mathbf{j} - 4\,\mathbf{k}$. [8 pts]

   Solution:

   We have:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{(2)(1) + (1)(5) + (-3)(-4)}{\sqrt{2^2 + 1^2 + (-3)^2}\sqrt{1^2 + 5^2 + (-4)^2}}$$

   (b) Find the distance between the point $(5, 12, -13)$ and the plane containing the points $(1, 1, 1)$, $(7, -1, -1)$, and $(0, 3, 0)$. [12 pts]

   Solution: To find a normal vector for the plane we first find two vectors parallel to the plane and cross them. To find $\mathbf{a}$ we go from the first point to the second and to find $\mathbf{b}$ we go from the first point to the third:

$$\begin{aligned} \mathbf{a} &= 6\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k} \\ \mathbf{b} &= -1\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{N} = \mathbf{a}\times\mathbf{b} &= 6\,\mathbf{i} + 8\,\mathbf{j} + 10\,\mathbf{k} \end{aligned}$$

   Then if we put $P = (1, 1, 1)$ and $Q = (5, 12, -13)$ then $\mathbf{PQ} = 4\,\mathbf{i} + 11\,\mathbf{j} - 14\,\mathbf{k}$ and the distance is

$$\frac{|\mathbf{PQ}\cdot\mathbf{N}|}{||\mathbf{N}||} = \frac{|(4)(6) + (11)(8) + (-14)(10)|}{\sqrt{6^2 + 8^2 + 10^2}}$$

2. Find the length of the curve $\mathbf{r}(t) = \frac{1}{2}t^2\,\mathbf{i} + \ln(t)\,\mathbf{j} + t\sqrt{2}\,\mathbf{k}$ from the point $\left(\frac{1}{2}, 0, \sqrt{2}\right)$ to the point $\left(2, \ln(2), 2\sqrt{2}\right)$. [20 pts]

   Solution: The portion of the curve is from $t = 1$ to $t = 2$. We have:

$$\begin{aligned} \mathbf{r}'(t) &= t\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + \sqrt{2}\,\mathbf{k} \\ ||\mathbf{r}'(t)|| &= \sqrt{t^2 + \left(\frac{1}{t}\right)^2 + (\sqrt{2})^2} \\ &= \sqrt{t^2 + \frac{1}{t^2} + 2} \\ &= \sqrt{\left(t + \frac{1}{t}\right)^2} \\ &= t + \frac{1}{t} \end{aligned}$$

and so

$$\begin{aligned}
\text{Length} &= \int_1^2 t + \frac{1}{t}\, dt \\
&= \frac{1}{2}t^2 + \ln|t| \Big|_1^2 \\
&= \left[\frac{1}{2}(2)^2 + \ln 2\right] - \left[\frac{1}{2}(1)^2 + \ln 1\right]
\end{aligned}$$

3. A particle with initial position $(0, 0, 1)$ has velocity $\mathbf{v}(t) = \frac{3}{2}(t+1)^{1/2}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \frac{1}{t+1}\,\mathbf{k}$ for any time t. Find the particle's acceleration and postition vectors, $\mathbf{a}(t)$ and $\mathbf{r}(t)$ at any time t. [20 pts]

Solution: First the acceleration:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{3}{4}(t+1)^{-1/2}\,\mathbf{i} - e^{-t}\,\mathbf{j} - \frac{1}{(t+1)^2}\,\mathbf{k}$$

Next:

$$\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{v}(t)\, dt \\
&= (t+1)^{3/2}\,\mathbf{i} - e^{-t}\,\mathbf{j} + \ln|t+1|\,\mathbf{k} + \mathbf{C} \\
\mathbf{r}(0) = 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} &= (0+1)^{3/2}\,\mathbf{i} - e^{-0}\,\mathbf{j} + \ln|0+1|\,\mathbf{k} + \mathbf{C} \\
0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} &= 1\,\mathbf{i} - 1\,\mathbf{j} + 0\,\mathbf{k} + \mathbf{C} \\
\mathbf{C} &= -1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}
\end{aligned}$$

So that

$$\mathbf{r}(t) = ((t+1)^{3/2} - 1)\,\mathbf{i} + (1 - e^{-t})\,\mathbf{j} + (\ln(t+1) + 1)\,\mathbf{k}$$

4. (a) Show that the line with parametrization $\mathbf{r}(t) = (2+t)\,\mathbf{i} + (1+t)\,\mathbf{j} + \left(\frac{1}{2} - \frac{1}{2}t\right)\,\mathbf{k}$ does not intersect the plane $x + 2y + 6z = 10$. [10 pts]

Solution: On the line we have $x = 2 + t$, $y = 1 + t$ and $z = \frac{1}{2} - \frac{1}{2}t$. For this to intersect the plane we would have:

$$\begin{aligned}
x + 2y + 6z &= 10 \\
(2+t) + 2(1+t) + 6\left(\frac{1}{2} - \frac{1}{2}t\right) &= 10 \\
2 + t + 2 + 2t + 3 - 3t &= 10 \\
7 &= 10
\end{aligned}$$

Therefore the line does not intersect the plane.

(b) Find the symmetric equation of the line containing the points $(1, 2, 0)$ and $(3, 6, 0)$. [10 pts]

Solution: The direction vector is

$$\mathbf{L} = 2\,\mathbf{i} + 4\,\mathbf{j} + 0\,\mathbf{k}$$

and so the final solution is:

$\frac{x-1}{2} = \frac{y-2}{4}$ and $z = 0$

5. Find and categorize all local maximum and minimum values and saddle points of: [20 pts]

$$f(x,y) = 3xy - x^2y - xy^2$$

Solution:

We have

$$f_x = 3y - 2xy - y^2 = y(3 - 2x - y)$$

$$f_y = 3x - x^2 - 2xy$$

The first tells us $y = 0$ or $y = 3 - 2x$.

In the first case the second then tells us $x = 0, 3$ yielding points $(0,0)$ and $(3,0)$.

In the second case the second then tells us that $x = 0, 1$ yielding points $(0,3)$ and $(1,1)$.

Next

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-2y)(-2x) - (3 - 2x - 2y)^2$$

So checking our points:

$D(0,0) = -$ so it's a saddle.

$D(3,0) = -$ so it's a saddle.

$D(0,3) = -$ so it's a saddle.

$D(1,1) = +$ and $f_{xx}(1,1) = -$ so it's a relative maximum.

6. A mountain's altitude is given by $h(x,y) = 2000 - 0.01x^2 - 0.005y^2$. You are standing on the mountain at the point $(50, 100, 1925)$.

   (a) You begin to walk in the direction of $\mathbf{a} = 3\,\mathbf{i} - 2\,\mathbf{j}$. Will your altitude be increasing or decreasing? [10 pts]

   Solution: First we have:

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{3}{\sqrt{13}}\,\mathbf{i} - \frac{2}{\sqrt{13}}\,\mathbf{j}$$

   Next

$$h_x = -0.02x \text{ and } h_y = -0.01y$$

   so that

$$D_{\mathbf{u}}h(50,100) = \frac{3}{\sqrt{13}}(-0.02(50)) - \frac{2}{\sqrt{13}}(-0.01(100)) = \frac{3(-1) + 2(1)}{\sqrt{13}} < 0$$

   so my altitude is decreasing.

(b) In which direction should you move so that your altitude increases the most rapidly? What is this rate of change? [10 pts]

Solution: We should head in the direction:

$$\nabla h = -0.02x\,\mathbf{i} - 0.01y\,\mathbf{j}$$
$$\nabla h(50, 100) = -1\,\mathbf{i} - 1\,\mathbf{j}$$

and the altitude will increase by

$$||\nabla h(50, 100)|| = \sqrt{(-1)^2 + (-1)^2}$$

7. (a) Find the volume of the solid enclosed by the cylinder $x^2 + y^2 = 4$ and between the planes $y + z = 2$ and $z = 0$. [12 pts]

Solution:

The solid is under the $z = 2 - y$ and above $z = 0$ within the region R which is the disk of radius 2 centered at the origin.

As a double integral

$$\begin{aligned}
\text{Volume} &= \iint_R 2 - y\,dA \\
&= \int_0^{2\pi} \int_0^2 (2 - r\sin\theta) r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 2r - r^2 \sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi} r^2 - \frac{1}{3} r^3 \sin\theta \Big|_0^2\,d\theta \\
&= \int_0^{2\pi} 4 - \frac{8}{3}\sin\theta\,d\theta \\
&= 4\theta + \frac{8}{3}\cos\theta \Big|_0^{2\pi} \\
&= \left[4(2\pi) + \frac{8}{3}(1)\right] - \left[0 + \frac{8}{3}(1)\right]
\end{aligned}$$

(b) Let D be the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant. Parametrize the following integral as an iterated integral in spherical coordinates but do not evaluate: [8 pts]

$$\iiint_D \sqrt{x^2 + y^2 + z^2}\,dV$$

Solution: The solution is

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

8. Evaluate the integral: [20 pts]

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx$$

Solution:

We need to change the order of integration. Picture omitted but the result is:

$$\begin{aligned}\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dy\, dx &= \int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\, dx\, dy \\ &= \int_0^2 \left.\frac{x}{y^3+1}\right|_0^{y^2} dy \\ &= \int_0^2 \frac{y^2}{y^3+1}\, dy \\ &= \left.\frac{1}{3}\ln|y^3+1|\right|_0^2 \\ &= \frac{1}{3}\ln(9) - \frac{1}{3}\ln(1)\end{aligned}$$

9. Let C be the intersection of the cylinder $(x-1)^2+y^2=1$ with the plane $x+y+z=4$ and with counterclockwise orientation when viewed from above. Use Stokes' Theorem to convert the line integral [20 pts]

$$\int_C xy\ dx + y\ dy + xz\ dz$$

to a surface integral. Parametrize the surface integral as an iterated integral in polar coordinates. Do not evaluate this integral.

Solution: The vector field is

$$\mathbf{F}(x,y,z) = xy\,\mathbf{i} + y\,\mathbf{j} + xz\,\mathbf{k}$$

so that

$$\nabla \times F = (0-0)\,\mathbf{i} - (z-0)\,\mathbf{j} + (0-x)\,\mathbf{k} = 0\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$$

and so

$$\int_C xy\ dx + y\ dy + xz\ dz = \iint_\Sigma (0\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

where Σ is the part of the plane $x+y+z=4$ inside the cylinder with "upwards" orientation.

We parametrize Σ:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (4 - r\cos\theta - r\sin\theta)\,\mathbf{k}$$
$$\text{with } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \text{ and } 0 \le r \le 2\cos\theta$$

Then:

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + (-\cos\theta - \sin\theta)\,\mathbf{k}\\ \mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} + (r\sin\theta - r\cos\theta)\,\mathbf{k}\\ \mathbf{r}_r \times \mathbf{r}_\theta &= r\,\mathbf{i} + r\,\mathbf{j} + r\,\mathbf{k}\end{aligned}$$

This cross product matches Σ's orientation and so:

$$\begin{aligned}\iint_\Sigma (0\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k})\cdot\mathbf{n}\,dS &= +\iint_R [0\,\mathbf{i} - (4 - r\cos\theta - r\sin\theta)\,\mathbf{j} - (r\cos\theta)\,\mathbf{k}]\cdot[r\,\mathbf{i} + r\,\mathbf{j} + r\,\mathbf{k}]\,dA\\ &= \int_{\pi/2}^{\pi/2}\int_0^{2\cos\theta} -r(4 - r\cos\theta - r\sin\theta) - r(r\cos\theta)\,dr\,d\theta\end{aligned}$$

10. (a) Use the Fundamental Theorem of Line Integrals to evaluate $\int_C 2x\ dx - z\ dy + (1-y)\ dz$ [10 pts] where C is the curve with parametrization $\mathbf{r}(t) = \sqrt{t+1}\,\mathbf{i} + \frac{1}{t+1}\,\mathbf{j} + e^t\,\mathbf{k}$ for $0 \le t \le 3$.

Solution:
The potential function is $f(x, y, z) = x^2 - yz + z$.
The start point is $\mathbf{r}(0) = 1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k}$ or $(1, 1, 1)$.
The end point is $\mathbf{r}(3) = 2\,\mathbf{i} + \frac{1}{4}\,\mathbf{j} + e^3\,\mathbf{k}$ or $(2, 1/4, e^3)$.
Therefore

$$\begin{aligned}\int_C 2x\ dx - z\ dy + (1-y)\ dz &= f(2, 1/4, e^3) - f(1, 1, 1)\\ &= \left[2^2 - (e^3)(1/4) + e^3\right] - \left[1^2 - (1)(1) + 1\right]\end{aligned}$$

(b) Use the Divergence Theorem to evaluate $\iint_\Sigma (2x\,\mathbf{i} + 3y\,\mathbf{j} + 3z\,\mathbf{k})\cdot\mathbf{n}\ dS$ where Σ is the [10 pts] sphere $x^2 + y^2 + z^2 = 5$ oriented inwards. That is, the unit normal vector $\mathbf{n}$ on Σ is directed inwards.

Solution: We see that Σ is the boundary of D, the solid sphere. Because of the inwards orientation we have:

$$\begin{aligned}
\iint\limits_{\Sigma} (2x\,\mathbf{i} + 3y\,\mathbf{j} + 3z\,\mathbf{k}) \cdot \mathbf{n}\, dS &= -\iiint\limits_{D} 2 + 3 + 3\, dV \\
&= -8 \iiint\limits_{1} dV \\
&= -8(\text{Volume of D}) \\
&= -8\left(\frac{4}{3}\pi \left(\sqrt{5}\right)^3\right)
\end{aligned}$$