Math 241 Spring 2017 Final Exam Solution

1. (a) Find \( \cos \theta \) where \( \theta \) is the angle between the vectors \( \mathbf{a} = 2\mathbf{i} + 1\mathbf{j} - 3\mathbf{k} \) and \( \mathbf{b} = 1\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} \). [8 pts]

Solution:
We have:

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} = \frac{(2)(1) + (1)(5) + (-3)(-4)}{\sqrt{2^2 + 1^2 + (-3)^2} \sqrt{1^2 + 5^2 + (-4)^2}}
\]

(b) Find the distance between the point \( (5, 12, -13) \) and the plane containing the points \( (1, 1, 1), (7, -1, -1), \) and \( (0, 3, 0) \). [12 pts]

Solution: To find a normal vector for the plane we first find two vectors parallel to the plane and cross them. To find \( \mathbf{a} \) we go from the first point to the second and to find \( \mathbf{b} \) we go from the first point to the third:

\[
\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \\
\mathbf{b} = -1\mathbf{i} + 2\mathbf{j} - 1\mathbf{k} \\
\mathbf{N} = \mathbf{a} \times \mathbf{b} = 6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}
\]

Then if we put \( P = (1,1,1) \) and \( Q = (5, 12, -13) \) then \( \mathbf{PQ} = 4\mathbf{i} + 11\mathbf{j} - 14\mathbf{k} \) and the distance is
\[
\frac{||\mathbf{PQ} \cdot \mathbf{N}||}{||\mathbf{N}||} = \frac{|(4)(6) + (11)(8) + (-14)(10)|}{\sqrt{6^2 + 8^2 + 10^2}}
\]

2. Find the length of the curve \( \mathbf{r}(t) = \frac{1}{2}t^2 \mathbf{i} + \ln(t) \mathbf{j} + t\sqrt{2} \mathbf{k} \) from the point \( \left( \frac{1}{2}, 0, \sqrt{2} \right) \) to the point \( (2, \ln(2), 2\sqrt{2}) \). [20 pts]

Solution: The portion of the curve is from \( t = 1 \) to \( t = 2 \). We have:

\[
\mathbf{r}'(t) = t \mathbf{i} + \frac{1}{t} \mathbf{j} + \sqrt{2} \mathbf{k} \\
||\mathbf{r}'(t)|| = \sqrt{t^2 + \left( \frac{1}{t} \right)^2 + (\sqrt{2})^2} = \sqrt{t^2 + \frac{1}{t^2} + 2} = \sqrt{\left( t + \frac{1}{t} \right)^2} = t + \frac{1}{t}
\]
and so

\[
\text{Length} = \int_1^2 t + \frac{1}{t} \, dt
\]

\[
= \left[ \frac{1}{2} t^2 + \ln |t| \right]_1^2
\]

\[
= \left[ \frac{1}{2} (2)^2 + \ln 2 \right] - \left[ \frac{1}{2} (1)^2 + \ln 1 \right]
\]

3. A particle with initial position \((0, 0, 1)\) has velocity \(\mathbf{v}(t) = \frac{3}{2} (t + 1)^{1/2} \mathbf{i} + e^{-t} \mathbf{j} + \frac{1}{t+1} \mathbf{k}\) for any \([20 \text{ pts}]\) time \(t\). Find the particle’s acceleration and position vectors, \(\mathbf{a}(t)\) and \(\mathbf{r}(t)\) at any time \(t\).

**Solution:** First the acceleration:

\[
\mathbf{a}(t) = \mathbf{v}'(t) = \frac{3}{4} (t + 1)^{-1/2} \mathbf{i} - e^{-t} \mathbf{j} - \frac{1}{(t+1)^2} \mathbf{k}
\]

Next:

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt
\]

\[
= (t + 1)^{3/2} \mathbf{i} - e^{-t} \mathbf{j} + \ln |t + 1| \mathbf{k} + \mathbf{C}
\]

\[
\mathbf{r}(0) = 0 \mathbf{i} + 0 \mathbf{j} + 1 \mathbf{k} = (0 + 1)^{3/2} \mathbf{i} - e^{-0} \mathbf{j} + \ln |0 + 1| \mathbf{k} + \mathbf{C}
\]

\[
0 \mathbf{i} + 0 \mathbf{j} + 1 \mathbf{k} = 1 \mathbf{i} - 1 \mathbf{j} + 0 \mathbf{k} + \mathbf{C}
\]

\[
\mathbf{C} = -1 \mathbf{i} + 1 \mathbf{j} + 1 \mathbf{k}
\]

So that

\[
\mathbf{r}(t) = ((t + 1)^{3/2} - 1) \mathbf{i} + (1 - e^{-t}) \mathbf{j} + (\ln(t + 1) + 1) \mathbf{k}
\]

4. (a) Show that the line with parametrization \(\mathbf{r}(t) = (2 + t) \mathbf{i} + (1 + t) \mathbf{j} + (\frac{1}{2} - \frac{1}{2} t) \mathbf{k}\) does not \([10 \text{ pts}]\) intersect the plane \(x + 2y + 6z = 10\).

**Solution:** On the line we have \(x = 2 + t, \ y = 1 + t\) and \(z = \frac{1}{2} - \frac{1}{2} t\). For this to intersect the plane we would have:

\[
x + 2y + 6z = 10
\]

\[
(2 + t) + 2(1 + t) + 6 \left( \frac{1}{2} - \frac{1}{2} t \right) = 10
\]

\[
2 + t + 2 + 2t + 3 - 3t = 10
\]

\[
7 = 10
\]

Therefore the line does not intersect the plane.

(b) Find the symmetric equation of the line containing the points \((1, 2, 0)\) and \((3, 6, 0)\). \([10 \text{ pts}]\)

**Solution:** The direction vector is

\[
\mathbf{L} = 2 \mathbf{i} + 4 \mathbf{j} + 0 \mathbf{k}
\]

and so the final solution is:
\[
\frac{x-1}{2} = \frac{y-2}{4} \quad \text{and} \quad z = 0
\]

5. Find and categorize all local maximum and minimum values and saddle points of: \[20 \text{ pts}\]

\[ f(x, y) = 3xy - x^2y - xy^2 \]

**Solution:**

We have

\[
x = 3y - 2xy - y^2 = y(3 - 2x - y)
\]

\[
y = 3x - x^2 - 2xy
\]

The first tells us \( y = 0 \) or \( y = 3 - 2x \).

In the first case the second then tells us \( x = 0 \), \( 3 \) yielding points \((0, 0)\) and \((3, 0)\).

In the second case the second then tells us that \( x = 0 \), \( 1 \) yielding points \((0, 3)\) and \((1, 1)\).

Next

\[
D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (-2y)(-2x) - (3 - 2x - 2y)^2
\]

So checking our points:

\[
D(0, 0) = - \text{so it’s a saddle.}
\]

\[
D(3, 0) = - \text{so it’s a saddle.}
\]

\[
D(0, 3) = - \text{so it’s a saddle.}
\]

\[
D(1, 1) = + \text{ and } f_{xx}(1, 1) = - \text{ so it’s a relative maximum.}
\]

6. A mountain’s altitude is given by \( h(x, y) = 2000 - 0.01x^2 - 0.005y^2 \). You are standing on the mountain at the point \((50, 100, 1925)\).

(a) You begin to walk in the direction of \( \mathbf{a} = 3 \mathbf{i} - 2 \mathbf{j} \). Will your altitude be increasing or decreasing? \[10 \text{ pts}\]

**Solution:** First we have:

\[
\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{3}{\sqrt{13}} \mathbf{i} - \frac{2}{\sqrt{13}} \mathbf{j}
\]

Next

\[
h_x = -0.02x \quad \text{and} \quad h_y = -0.01y
\]

so that

\[
D_{\mathbf{u}} h(50, 100) = \frac{3}{\sqrt{13}} (-0.02(50)) - \frac{2}{\sqrt{13}} (-0.01(100)) = \frac{3(-1) + 2(1)}{\sqrt{13}} < 0
\]

so my altitude is decreasing.
(b) In which direction should you move so that your altitude increases the most rapidly? What is this rate of change?

**Solution:** We should head in the direction:

\[
\nabla h = -0.02x \mathbf{i} - 0.01y \mathbf{j}
\]

\[
\nabla h(50, 100) = -1 \mathbf{i} - 1 \mathbf{j}
\]

and the altitude will increase by

\[
||\nabla h(50, 100)|| = \sqrt{(-1)^2 + (-1)^2}
\]

7. (a) Find the volume of the solid enclosed by the cylinder \(x^2 + y^2 = 4\) and between the planes \(y + z = 2\) and \(z = 0\). [12 pts]

**Solution:**

The solid is under the \(z = 2 - y\) and above \(z = 0\) within the region \(R\) which is the disk of radius 2 centered at the origin.

As a double integral

\[
\text{Volume} = \int \int_R 2 - y \, dA
\]

\[
= \int_0^{2\pi} \int_0^2 (2 - r \sin \theta) r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 2r - r^2 \sin \theta \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left[ r^2 - \frac{1}{3} r^3 \sin \theta \right]_0^2 \, d\theta
\]

\[
= \int_0^{2\pi} 4 - \frac{8}{3} \sin \theta \, d\theta
\]

\[
= 4\theta + \frac{8}{3} \cos \theta \Bigg|_0^{2\pi}
\]

\[
= \left[ 4(2\pi) + \frac{8}{3}(1) \right] - \left[ 0 + \frac{8}{3}(1) \right]
\]

(b) Let \(D\) be the solid between the spheres \(x^2 + y^2 + z^2 = 1\) and \(x^2 + y^2 + z^2 = 4\) in the first octant. Parametrize the following integral as an iterated integral in spherical coordinates but do not evaluate:

\[
\int \int \int_D \sqrt{x^2 + y^2 + z^2} \, dV
\]

**Solution:** The solution is

\[
\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
8. Evaluate the integral:  
\[ \int_0^4 \int_{\sqrt{x}}^{\sqrt{y}} \frac{1}{y^3 + 1} \, dy \, dx \]  

Solution:
We need to change the order of integration. Picture omitted but the result is:
\[ \int_0^4 \int_{\sqrt{x}}^{\sqrt{y}} \frac{1}{y^3 + 1} \, dy \, dx = \int_0^2 \int_{\sqrt{x}}^{y^2} \frac{1}{y^3 + 1} \, dx \, dy = \int_0^2 \frac{x}{y^2} \left[ \frac{1}{y^3 + 1} \right]_0^y \, dy = \int_0^2 \frac{1}{3} \ln |y^3 + 1|_0^2 = \frac{1}{3} \ln(9) - \frac{1}{3} \ln(1) \]

9. Let \( C \) be the intersection of the cylinder \((x-1)^2 + y^2 = 1\) with the plane \(x + y + z = 4\) and with counterclockwise orientation when viewed from above. Use Stokes’ Theorem to convert the line integral
\[ \int_C xy \, dx + y \, dy + xz \, dz \]
to a surface integral. Parametrize the surface integral as an iterated integral in polar coordinates. Do not evaluate this integral.

Solution: The vector field is
\[ \mathbf{F}(x, y, z) = xy \mathbf{i} + y \mathbf{j} + xz \mathbf{k} \]
so that
\[ \nabla \times F = (0 - 0) \mathbf{i} - (z - 0) \mathbf{j} + (0 - x) \mathbf{k} = 0 \mathbf{i} - z \mathbf{j} - x \mathbf{k} \]
and so
\[ \int_C xy \, dx + y \, dy + xz \, dz = \iint_{\Sigma} (0 \mathbf{i} - z \mathbf{j} - x \mathbf{k}) \cdot \mathbf{n} \, dS \]
where \( \Sigma \) is the part of the plane \(x + y + z = 4\) inside the cylinder with "upwards" orientation. We parametrize \( \Sigma \):
\[ \mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (4 - r \cos \theta - r \sin \theta) \mathbf{k} \]
with \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) and \(0 \leq r \leq 2 \cos \theta\)
Then:

\[ \mathbf{r}_r = \cos \theta \, \mathbf{i} + \sin \theta \, \mathbf{j} + (-\cos \theta - \sin \theta) \, \mathbf{k} \]
\[ \mathbf{r}_\theta = -r \sin \theta \, \mathbf{i} + r \cos \theta \, \mathbf{j} + (r \sin \theta - r \cos \theta) \, \mathbf{k} \]
\[ \mathbf{r}_r \times \mathbf{r}_\theta = r \, \mathbf{i} + r \, \mathbf{j} + r \, \mathbf{k} \]

This cross product matches \( \Sigma \)'s orientation and so:

\[
\iint_{\Sigma} (0 \mathbf{i} - z \mathbf{j} - x \mathbf{k}) \cdot \mathbf{n} \, dS = + \iint_{R} [0 \mathbf{i} - (4 - r \cos \theta - r \sin \theta) \mathbf{j} - (r \cos \theta) \mathbf{k}] \cdot [r \mathbf{i} + r \mathbf{j} + r \mathbf{k}] \, dA \\
= \int_{\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} -r(4 - r \cos \theta - r \sin \theta) - r(r \cos \theta) \, dr \, d\theta
\]

10. (a) Use the Fundamental Theorem of Line Integrals to evaluate \( \int_{C} 2x \, dx - z \, dy + (1 - y) \, dz \) \[ 10 \text{ pts} \]

where \( C \) is the curve with parametrization \( \mathbf{r}(t) = \sqrt{t+1} \, \mathbf{i} + \frac{1}{t+1} \, \mathbf{j} + e^t \, \mathbf{k} \) for \( 0 \leq t \leq 3 \).

**Solution:**

The potential function is \( f(x, y, z) = x^2 - yz + z \).

The start point is \( \mathbf{r}(0) = 1 \, \mathbf{i} + 1 \, \mathbf{j} + 1 \, \mathbf{k} \) or \( (1, 1, 1) \).

The end point is \( \mathbf{r}(3) = 2 \, \mathbf{i} + \frac{1}{3} \, \mathbf{j} + e^3 \, \mathbf{k} \) or \( (2, 1/4, e^3) \).

Therefore

\[
\int_{C} 2x \, dx - z \, dy + (1 - y) \, dz = f(2, 1/4, e^3) - f(1, 1, 1) \\
= [2^2 - (e^3)(1/4) + e^3] - [1^2 - (1)(1) + 1]
\]

(b) Use the Divergence Theorem to evaluate \( \iint_{\Sigma} (2x \, \mathbf{i} + 3y \, \mathbf{j} + 3z \, \mathbf{k}) \cdot \mathbf{n} \, dS \) where \( \Sigma \) is the \[ 10 \text{ pts} \]

sphere \( x^2 + y^2 + z^2 = 5 \) oriented inwards. That is, the unit normal vector \( \mathbf{n} \) on \( \Sigma \) is directed inwards.

**Solution:** We see that \( \Sigma \) is the boundary of \( D \), the solid sphere. Because of the inwards orientation we have:
$$\int \int \int \Sigma (2x \mathbf{i} + 3y \mathbf{j} + 3z \mathbf{k}) \cdot \mathbf{n} dS = -\int \int \int_D 2 + 3 + 3 dV$$

$$= -8 \int \int \int_1 dV$$

$$= -8 \text{(Volume of D)}$$

$$= -8 \left( \frac{4}{3} \pi (\sqrt{5})^3 \right)$$