Math 241 Fall 2018 Final Exam Solution

1. Parts (a) and (b) are independent.

(a) Use the dot product to show that the three points \( A = (-2, 3, -4) \), \( B = (0, 11, -1) \), and \( C = (1, 10, -5) \) form a right triangle. Then find the length of the hypotenuse.

Solution: We have:

\[
\overrightarrow{AB} = 2\mathbf{i} + 8\mathbf{j} + 3\mathbf{k} \\
\overrightarrow{AC} = 3\mathbf{i} + 7\mathbf{j} - 1\mathbf{k} \\
\overrightarrow{BC} = 1\mathbf{i} - 1\mathbf{j} - 4\mathbf{k}
\]

And so:
\[
\overrightarrow{AB} \cdot \overrightarrow{AC} \neq 0 \\
\overrightarrow{AB} \cdot \overrightarrow{BC} \neq 0 \\
\overrightarrow{AC} \cdot \overrightarrow{BC} = 0
\]

So we have a right triangle. The length of the hypotenuse is then
\[
|\overrightarrow{AB}| = \sqrt{2^2 + 8^2 + 3^2}
\]

(b) Find the distance between the point \( Q = (-2, 1, 3) \) and the line \( x = 2, \ \frac{y+3}{2} = z - 4 \).

Solution: The point \( P = (2, -3, 4) \) is on the line and \( \mathbf{L} = 0\mathbf{i} - 2\mathbf{j} + 1\mathbf{k} \). We have \( \overrightarrow{PQ} = -4\mathbf{i} + 4\mathbf{j} - 1\mathbf{k} \) so that

\[
\text{dist} = \frac{||\overrightarrow{PQ} \times \mathbf{L}||}{||\mathbf{L}||} = \frac{||2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}||}{||0\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}||} = \frac{\sqrt{4 + 16 + 64}}{\sqrt{0 + 4 + 1}}
\]
2. Parts (a) and (b) are independent.

(a) An 80 pound force and a 50 pound force are applied to an object at the same point with an angle of $\frac{\pi}{6}$ between them. Find the magnitude of the resultant force on the object.

Solution: If we put the object at the origin and the 50lb force on the positive $x$ axis and the 80lb force in the first quadrant then $F_1 = 50 \mathbf{i} + 0 \mathbf{j}$ and $F_2 = 80 \cos(\pi/6) \mathbf{i} + 80 \sin(\pi/6) \mathbf{j} = 40\sqrt{3} \mathbf{i} + 40 \mathbf{j}$ and so

$$||F_1 + F_2|| = \sqrt{(50 + 40\sqrt{3})^2 + (40)^2}$$

(b) Show that the two lines $r_1(t) = (t + 1) \mathbf{i} + 2t \mathbf{j} + 3 \mathbf{k}$ and $r_2(s) = s \mathbf{i} + (4 - s) \mathbf{j} + (s + 1) \mathbf{k}$ intersect and are not parallel, then find the equation of the plane containing them.

Solution: The lines meet when $t + 1 = s$, $2t = 4 - s$ and $3 = s + 1$. The last gives $s = 2$ and so $t = 1$.

Noting $r_1(1) = 2 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k}$ they meet at $(2, 2, 3)$.

The vectors are $L_1 = 1 \mathbf{i} + 2 \mathbf{j} + 0 \mathbf{k}$ and $L_2 = 1 \mathbf{i} - 1 \mathbf{j} + 1 \mathbf{k}$ and these are not multiples so the lines are not parallel.

We use $n = L_1 \times L_2 = 2 \mathbf{i} - 1 \mathbf{j} - 3 \mathbf{k}$ and so the plane equation is

$$2(x - 2) - 1(y - 2) - 3(z - 3) = 0$$
3. Consider the curve \( C \) parametrized by \( \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j} \).

(a) Find the tangent vector \( \mathbf{T}(t) \). Simplify. [10 pts]

**Solution:** We have:

\[
\mathbf{r}'(t) = (e^t \cos t - e^t \sin t) \mathbf{i} + (e^t \sin t + e^t \cos t) \mathbf{j}
\]

\[
||\mathbf{r}'(t)|| = \sqrt{e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t + e^{2t} \cos^2 t + 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t} = \sqrt{2e^{2t}} = e^t \sqrt{2}
\]

and so

\[
\mathbf{T}(t) = \frac{1}{\sqrt{2}} \cos t \mathbf{i} + \frac{1}{\sqrt{2}} \sin t \mathbf{j}
\]

(b) Find the normal vector \( \mathbf{N}(t) \). [10 pts]

**Solution:** We have:

\[
\mathbf{T}'(t) = -\frac{1}{\sqrt{2}} \sin t \mathbf{i} + \frac{1}{\sqrt{2}} \cos t \mathbf{j}
\]

\[
||\mathbf{T}'(t)|| = \sqrt{\frac{1}{2} \sin^2 t + \frac{1}{2} \cos^2 t} = \frac{1}{\sqrt{2}}
\]

and so

\[
\mathbf{N}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}
\]
4. Let $C$ be the curve defined as the portion of the parabola $y = x^2$ in the plane $z = -2$ between the points $(2, 4, -2)$ and $(3, 9, -2)$.

(a) Find a parametrization of $C$.  
   **Solution:** We have $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} - 2 \mathbf{k}$ for $2 \leq t \leq 3$.

(b) Set up the iterated integral that computes the length of $C$.  **Do not evaluate the integral!**  
   **Solution:** We have
   \[
   \mathbf{r}'(t) = 1 \mathbf{i} + 2t \mathbf{j} + 0 \mathbf{k}
   \]
   \[
   ||\mathbf{r}'(t)|| = \sqrt{1 + 4t^2}
   \]
   Length $= \int_{2}^{3} \sqrt{1 + 4t^2} \, dt$
5. Use Lagrange Multipliers to find the extreme values of $f(x, y) = 3x - y$ subject to the constraint $x^2 + 2y^2 = 1$. You may assume these values exist.

**Solution:** We set $g(x, y) = x^2 + 2y^2$ and then solve the system

$$
3 = \lambda 2x \\
-1 = \lambda 4y \\
x^2 + 2y^2 = 1
$$

We can’t have $x = 0$ or $y = 0$ since these would contradict the first and second, therefore the first tells us $\lambda = \frac{3}{2x}$ and the second tells us $\lambda = -\frac{1}{4y}$. Thus

$$
\frac{3}{2x} = -\frac{1}{4y} \\
12y = -2x \\
x = -6y
$$

Plugging this into the third tells us

$$
36y^2 + 2y^2 = 1 \\
y = \pm 1/\sqrt{38}
$$

This yields the points $(-6/\sqrt{38}, 1/\sqrt{38})$ and $(6/\sqrt{38}, -1/\sqrt{38})$. Then:

$$
f(-6/\sqrt{38}, 1/\sqrt{38}) = -19/\sqrt{38} \text{ Min} \\
f(6/\sqrt{38}, -1/\sqrt{38}) = 19/\sqrt{38} \text{ Max}
$$
6. If \( f(x, y) = x^2y + xy + y^3 \) use tangent plane approximation at \((1, 2)\) to approximate \( f(0.95, 2.1) \). [20 pts]

**Solution:** We have

\[
\begin{align*}
    f_x &= 2xy + y \\
    f_y &= x^2 + x + 3y^2
\end{align*}
\]

and so

\[
\begin{align*}
    f(1, 2) &= 12 \\
    f_x(1, 2) &= 6 \\
    f_y(1, 2) &= 14
\end{align*}
\]

Therefore

\[
\begin{align*}
    f(0.95, 2.1) &\approx f(1, 2) + f_x(1, 2)(0.95 - 1) + f_y(1, 2)(2.1 - 2) \\
    &\approx 12 + 6(0.95 - 1) + 14(2.1 - 2)
\end{align*}
\]
7. Let $D$ be the solid that lies inside the sphere $x^2+y^2+z^2 = 2$ and outside the cylinder $x^2+y^2 = 1$.

(a) Set up an iterated triple integral in spherical coordinates that evaluates the volume of $D$. [10 pts]

Do not evaluate the integral!

Solution: In spherical the equations are $\rho = \sqrt{2}$ and $\rho^2 \sin^2 \phi = 1$ (or $\rho \sin \phi = 1$ or $\rho = \csc \phi$) respectively. These meet when $\sqrt{2} \sin \phi = 1$ or $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$. Therefore the volume is given by

$$\int_{0}^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{csc \phi}^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

(b) Set up an iterated triple integral in cylindrical coordinates that evaluates $\iiint_D x^2 \, dV$. [10 pts]

Do not evaluate the integral!

Solution: In cylindrical the equations of the sphere yields $z = \pm \sqrt{2-r^2}$ and so:

$$\iiint_D x^2 \, dV = \int_{0}^{2\pi} \int_{1}^{\sqrt{2}} \int_{-\sqrt{2-r^2}}^{+\sqrt{2-r^2}} (r \cos \theta)^2 r \, dz \, dr \, d\theta$$
8. Let $R$ be the region enclosed by the ellipse given by $9x^2 + y^2 = 9$. Using an appropriate change of variables, evaluate $\int\int_R x^2 \, dA$. Make sure you specify the change of variables, and draw the new region. **Evaluate the integral!**

**Solution:** Set $u = 3x$ and $v = y$ and then the ellipse becomes $u^2 + v^2 = 9$. We have $x = \frac{1}{3}u$ and $y = v$ and so the Jacobian of the change of variables is

$$
\begin{vmatrix}
\frac{1}{3} & 0 \\
0 & 1
\end{vmatrix} = \frac{1}{3}
$$

and so:

$$
\int\int_R x^2 \, dA = \int\int_S \frac{1}{9} u^2 |1/3| \, dA
$$

We rewrite this in polar:

$$
\int\int_S \frac{1}{9} u^2 |1/3| \, dA = \frac{1}{27} \int_0^{2\pi} \int_0^3 r^2 \cos^2 \theta \, r \, dr \, d\theta
$$

$$
= \frac{1}{27} \int_0^{2\pi} \frac{1}{4} r^4 \cos^2 \theta \bigg|_0^3 \, d\theta
$$

$$
= \frac{3}{4} \int_0^{2\pi} \cos^2 \theta \, d\theta
$$

$$
= \frac{3}{4} \int_0^{2\pi} \frac{1}{2} (1 + \cos(2\theta)) \, d\theta
$$

$$
= \frac{3}{8} \int_0^{2\pi} \left(\frac{1}{2} \sin(2\theta)\right) \left(\frac{1}{2} \cos(2\theta)\right) \, d\theta
$$

$$
= \frac{3}{8} \left(\frac{1}{2} \sin(2\theta)\right) \bigg|_0^{2\pi}
$$

$$
= \frac{3}{8} (2\pi)
$$
9. Let \( \Sigma \) be the portion of the plane \( z = 9 - x \) inside the cylinder \( r = 2 \cos \theta \). Let \( R \) be the edge of \( \Sigma \) with counterclockwise orientation when viewed from above. Use Stokes’ Theorem to rewrite the integral \( \int_C 2x \, dx + yz \, dy + x^2 z^2 \, dz \) as a surface integral, parametrize the surface and then proceed until you have an iterated double integral. Do not evaluate the integral!

**Solution:** Stokes’ Theorem tells us that
\[
\int_C 2x \, dx + yz \, dy + x^2 z^2 \, dz = \int \int \Sigma \left[ (0 - y) \mathbf{i} - (2xz^2 - 0) \mathbf{j} + (0 - 0) \mathbf{k} \right] \cdot \mathbf{n} \, dS
\]
where \( \Sigma \) is the part of the plane inside the cylinder with upwards orientation. We parametrize \( \Sigma \) as:
\[
r(r, \theta) = r \cos \theta \, \mathbf{i} + r \sin \theta \, \mathbf{j} + (9 - r \cos \theta) \, \mathbf{k}
\]
\[
-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
\]
\[
0 \leq r \leq 2 \cos \theta
\]
Then
\[
\mathbf{r}_r = \cos \theta \, \mathbf{i} + \sin \theta \, \mathbf{j} - \cos \theta \, \mathbf{k}
\]
\[
\mathbf{r}_\theta = -r \sin \theta \, \mathbf{i} + r \cos \theta \, \mathbf{j} + r \sin \theta \, \mathbf{k}
\]
\[
\mathbf{r}_r \times \mathbf{r}_\theta = r \, \mathbf{i} - 0 \, \mathbf{j} + r \, \mathbf{k}
\]
Since this matches \( \Sigma \)’s orientation the integral becomes
\[
\int \int \Sigma \left[ (0 - y) \mathbf{i} - (2xz^2 - 0) \mathbf{j} + (0 - 0) \mathbf{k} \right] \cdot \mathbf{n} \, dS
\]
\[
= + \int \int_R \left[ -r \sin \theta \, \mathbf{i} - 2(r \cos \theta)(9 - r \cos \theta)^2 \, \mathbf{j} + 0 \, \mathbf{k} \right] \cdot [r \, \mathbf{i} - 0 \, \mathbf{j} + r \, \mathbf{k}] \, dA
\]
\[
= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} -r^2 \sin \theta \, dr \, d\theta
\]
10. Parts (a) and (b) are independent.

(a) Use Green’s Theorem to evaluate \( \int_C xy \, dx + x \, dy \) where \( C \) is the triangle with vertices \((0,0), (3,0), \) and \((3,6)\), oriented clockwise. **Evaluate the integral!**

**Solution:** If \( R \) is the region inside the triangle then because of the orientation we have

\[
\int_C xy \, dx + x \, dy = - \int_R 1 - x \, dA
\]

\[
= - \int_0^3 \int_0^{2x} 1 - x \, dy \, dx
\]

\[
= - \int_0^3 \left[ y - xy \right]_0^{2x} \, dx
\]

\[
= - \int_0^3 2x - x(2x) \, dx
\]

\[
= - \int_0^3 x^2 - \frac{2}{3} x^3 \, dx
\]

\[
= -(3^2 - \frac{2}{3} (3)^3)
\]

(b) Let \( \Sigma \) be the portion of the paraboloid \( z = 16 - x^2 - y^2 \) restricted by \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 3 \). Write down an iterated double integral for the surface area of \( \Sigma \). **Do not evaluate the integral!**

**Solution:** The surface is parametrized by

\[
r(x, y) = x \mathbf{i} + y \mathbf{j} + (16 - x^2 - y^2) \mathbf{k}
\]

\[
0 \leq x \leq 2
\]

\[
0 \leq y \leq 3
\]

So we have

\[
r_x = 1 \mathbf{i} + 0 \mathbf{j} - 2x \mathbf{k}
\]

\[
r_y = 0 \mathbf{i} + 1 \mathbf{j} - 2y \mathbf{k}
\]

\[
r_x \times r_y = 2x \mathbf{i} + 2y \mathbf{j} + 1 \mathbf{k}
\]

\[
||r_x \times r_y|| = \sqrt{4x^2 + 4y^2 + 1}
\]

and so the surface area is

\[
\iint_{\Sigma} 1 \, dS = \int_R \sqrt{4x^2 + 4y^2 + 1} \, dA
\]

\[
= \int_0^2 \int_0^3 \sqrt{4x^2 + 4y^2 + 1} \, dy \, dx
\]