1. Parts (a) and (b) are independent.

   (a) Let $P = (-1, 3)$, $Q = (-2, 1)$, and $R = (1, -4)$. Find the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{QR}$. [10 pts]

   Solution:
   We have $\overrightarrow{PQ} = -1\,\mathbf{i} - 2\,\mathbf{j}$ and $\overrightarrow{QR} = 3\,\mathbf{i} - 5\,\mathbf{j}$ and so

$$\text{Proj}_{\overrightarrow{QR}}\overrightarrow{PQ} = \frac{(-1\,\mathbf{i} - 2\,\mathbf{j}) \cdot (3\,\mathbf{i} - 5\,\mathbf{j})}{(3\,\mathbf{i} - 5\,\mathbf{j}) \cdot (3\,\mathbf{i} - 5\,\mathbf{j})}(3\,\mathbf{i} - 5\,\mathbf{j}) = \frac{7}{34}(3\,\mathbf{i} - 5\,\mathbf{j})$$

   (b) Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to an object located at the origin in the xy-plane. The force $\mathbf{F}_1$ has a magnitude of 60 and makes an angle of $\frac{\pi}{6}$ with the positive x-axis and force $\mathbf{F}_2$ has a magnitude of 100 and makes an angle of $\frac{3\pi}{4}$ with the positive x-axis. Find the magnitude and direction (angle from positive x-axis) of the sum of these forces. [10 pts]

   Solution:
   The forces are

$$\mathbf{F}_1 = 60\cos\frac{\pi}{6}\,\mathbf{i} + 60\sin\frac{\pi}{6}\,\mathbf{j} = 30\sqrt{3}\,\mathbf{i} + 30\,\mathbf{j}$$

   and

$$\mathbf{F}_2 = 100\cos\frac{3\pi}{4}\,\mathbf{i} + 100\sin\frac{3\pi}{4} = -50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j}$$

   and so

$$\mathbf{F}_1 + \mathbf{F}_2 = (30\sqrt{3} - 50\sqrt{2})\,\mathbf{i} + (30 + 30\sqrt{2})\,\mathbf{j}$$

   The magnitude is then:

$$\sqrt{(30\sqrt{3} - 50\sqrt{2})^2 + (30 + 30\sqrt{2})^2}$$

   and the angle with the positive x-axis is:

$$\tan^{-1}\left(\frac{30 + 30\sqrt{2}}{30\sqrt{3} - 50\sqrt{2}}\right)$$

2. Parts (a) and (b) are independent.

   (a) Find the distance from the point $(1, -3, 2)$ to the line containing the points $(1, 0, 1)$ and $(5, 2, 0)$. [10 pts]

   Solution: The line has $\mathbf{L} = 4\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k}$. If $P = (1, 0, 1)$ is on the line and $Q = (1, -3, 2)$ is off the line then the distance is:

$$\begin{aligned}
\frac{||\overrightarrow{PQ} \times \mathbf{L}||}{||\mathbf{L}||} &= \frac{||(0\,\mathbf{i} - 3\,\mathbf{j} + 1\,\mathbf{k}) \times (4\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k})||}{||4\,\mathbf{i} + 2\,\mathbf{j} - 1\,\mathbf{k}||} \\
&= \frac{||1\,\mathbf{i} + 4\,\mathbf{j} + 12\,\mathbf{k}||}{\sqrt{16 + 4 + 1}} \\
&= \frac{\sqrt{1 + 16 + 144}}{\sqrt{16 + 4 + 1}}
\end{aligned}$$

(b) Find an equation of the form $ax + by + cz = d$ of the plane that contains the point $(1, 2, 3)$ and the line [10 pts]

$$\frac{x}{2} = \frac{y+3}{7} = \frac{z-4}{5}.$$

Solution:

The line has direction vector $\mathbf{L} = 2\,\mathbf{i} + 7\,\mathbf{j} + 5\,\mathbf{k}$ parallel to the plane and the vector $\mathbf{a} = -1\,\mathbf{i} - 5\,\mathbf{j} + 1\,\mathbf{k}$ joining $(1, 2, 3)$ to $(0, -3, 4)$ is also parallel to the plane. Thus:

$$\mathbf{N} = (2\,\mathbf{i} + 7\,\mathbf{j} + 5\,\mathbf{k}) \times (-1\,\mathbf{i} - 5\,\mathbf{j} + 1\,\mathbf{k}) = 32\,\mathbf{i} - 7\,\mathbf{j} - 3\,\mathbf{k}$$

and so the plane is

$$\begin{aligned} 32(x-1) - 7(y-2) - 3(z-3) &= 0 \\ 32x - 32 - 7y + 14 - 3z + 9 &= 0 \\ 32x - 7y - 3z &= 9 \end{aligned}$$

3. Parts (a) and (b) are independent.

(a) Find a parametrization of the quarter circle in the plane $z = 1$ with endpoints $(-1, 0, 1)$ and $(0, -1, 1)$ and center $(0, 0, 1)$. [8 pts]

Solution:

The most obvious answer might be:

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 1\,\mathbf{k} \text{ for } \pi \leq t \leq \tfrac{3\pi}{2}$$

(b) Determine if the following pair of lines is parallel, intersecting, or neither. If the lines intersect, find the point at which they intersect. [12 pts]

$$\mathbf{r}_1(t) = (1 + 6t)\mathbf{i} + (3 - 7t)\mathbf{j} + (2 + t)\mathbf{k}$$

$$\mathbf{r}_2(s) = (10 + 3s)\mathbf{i} + (6 + s)\mathbf{j} + (14 + 4s)\mathbf{k}$$

Solution:

In order to meet we would need:

$$\begin{aligned} 1 + 6t &= 10 + 3s \\ 3 - 7t &= 6 + s \\ 2 + t &= 14 + 4s \end{aligned}$$

The second states that $s = -7t - 3$ so if we plug into the third we get $2 + t = 14 + 4(-7t - 3)$ which yields $t = 0$ and so $s = -3$. These satisfy the first so they meet.

Since they meet at $t = 0$ this is the point $\mathbf{r}_1(0) = 1\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$ or $(1, 3, 2)$.

4. Find all critical points of $f(x,y) = x^3 - 6x^2 - 5y^2$ and classify each as a relative minimum, maximum, or saddle point. [20 pts]

Solution:

The critical points occur when

$$f_x = 3x^2 - 12x = 3x(x-4) = 0$$
$$f_y = -10y = 0$$

Thus $y = 0$ and $x = 0, 4$ yielding $(0,0)$ and $(4,0)$.

We have $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (6x-12)(-10) - (0)^2$ and so:

$D(0,0) = +$ and $f_{yy}(0,0) = -$ so relative maximum.

$D(4,0) = -$ so saddle point.

5. Use Lagrange Multipliers to find the maximum and minimum of $f(x,y) = x^3 - 6x^2 - 5y^2$ subject to the constraint $x^2 + y^2 = 1$. [20 pts]
Note: Your system will have six solutions.

Solution:

We solve the system:

$$3x^2 - 12x = \lambda 2x$$
$$-10y = \lambda 2y$$
$$x^2 + y^2 = 1$$

The first tells us that $x = 0$ or $\lambda = \frac{3x-12}{2}$. If $x = 0$ then the third tells us $y = \pm 1$ so $(0, \pm 1)$.

The second tells us that $y = 0$ or $\lambda = -5$. If $y = 0$ then the third tells us $x = \pm 1$ so $(\pm 1, 0)$.

Otherwise $\frac{3x-12}{2} = -5$ and so $3x = 2$ so $x = \frac{2}{3}$ so the third tells us that $y = \pm\sqrt{5/9}$ so $\left(2/3, \pm\sqrt{5/9}\right)$.

Then:

- $f(0,+1) = -5$
- $f(0,-1) = -5$
- $f(+1,0) = -5$
- $f(-1,0) = -7$
- $f(2/3, +\sqrt{5/9}) = \frac{8}{27} - 6\left(\frac{4}{9}\right) - 5\left(\frac{5}{9}\right) = -\frac{139}{27}$
- $f(2/3, -\sqrt{5/9}) = \frac{8}{27} - 6\left(\frac{4}{9}\right) - 5\left(\frac{5}{9}\right) = -\frac{139}{27}$

Noting $-5 = -\frac{135}{27}$ and $-7 = -\frac{189}{27}$ we see that the maximum is -5 and the minimum is -7.

6. Let R be the region in the first quadrant of the xy-plane bounded by the lines $y = x$ and $y = 3x$ and by the hyperbolas $xy = 1$ and $xy = 5$. Use the change of variables $x = \frac{u}{v}$ and $y = v$ to set [20 pts]

up an iterated double integral in the uv-plane representing the area of R.
Do not evaluate this integral.

Solution:

We have the new region S defined by:

$$\begin{aligned} y = x &\to v = u/v \to v = \sqrt{u} \\ y = 3x &\to v = 3u/v \to v = \sqrt{3u} \\ xy = 1 &\to u = 1 \\ xy = 5 &\to u = 5 \end{aligned}$$

We have:

$$J = \det \begin{bmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{bmatrix} = 1/v$$

Therefore:

$$\text{Area} = \int\int_R 1\, dA = \int\int_S |1/v|\, dA = \int_1^5 \int_{\sqrt{u}}^{\sqrt{3u}} 1/v\, dv\, du$$

7. Parts (a) and (b) are independent.

(a) Find the unit vector direction of maximum increase of $f(x, y) = x^2y - xy$ at the point $(2, -3)$. [10 pts]

Solution:

The direction of maximum increase is:

$$\begin{aligned} \nabla f(x, y) &= (2xy - y)\,\mathbf{i} + (x^2 - x)\,\mathbf{j} \\ \nabla f(2, -3) &= (2(2)(-3) - (-3))\,\mathbf{i} + ((2)^2 - 2)\,\mathbf{j} \\ &= -9\,\mathbf{i} + 2\,\mathbf{j} \end{aligned}$$

Therefore the corresponding unit vector is:

$$\frac{-9\,\mathbf{i} + 2\,\mathbf{j}}{\sqrt{(-9)^2 + 4^2}}$$

(b) Use tangent plane approximation to approximate $(16.05)^{1/4}(7.95)^{2/3}$. [10 pts]

Solution:

We set $f(x, y) = x^{1/4}y^{2/3}$ and we wish to approximate $f(16.05, 7.95)$. We use the anchor point $(16, 8)$ and then since:

$$
\begin{aligned}
f(x,y) &\approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\\
f(x,y) &\approx x_0^{1/4}y_0^{2/3} + \frac{1}{4}x_0^{-3/4}y_0^{2/3}(x-x_0) + \frac{2}{3}x_0^{1/4}y_0^{-1/3}(y-y_0)\\
f(16.05, 7.95) &\approx 16^{1/4}8^{2/3} + \frac{1}{4}(16)^{-3/4}(8)^{2/3}(16-16.05) + \frac{2}{3}(16)^{1/4}(8)^{-1/3}(8-7.95)
\end{aligned}
$$

8. Write down the iterated triple integral representing the volume of the solid region D bounded between the spheres (centered at the origin) of radius 1 and 3, and the upper part of the cone $z^2 = 3(x^2+y^2)$. [15 pts]
Do not evaluate this integral.

Solution:

We have:

$$\int_0^{2\pi}\int_0^{\pi/6}\int_1^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

9. Let C be the intersection of the parabolic sheet $y = 16 - x^2$ with the cylinder $x^2 + z^2 = 4$ with counterclockwise orientation when viewed from the positive y-axis. Use Stokes' Theorem to convert the line integral [25 pts]

$$\int_C x^2 z \, dx + x \, dy + yz \, dz$$

to a surface integral. Parametrize the surface to obtain a double iterated integral.
Do not evaluate this integral.

Solution:

By Stokes' Theorem we have:

$$\int_C x^2 z \, dx + x \, dy + yz \, dz = \int\int_\Sigma (z\,\mathbf{i} + x^2\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n} \, dS$$

where Σ is the portion of the parabolic sheet inside the cylinder, oriented in the positive y-direction.

We parametrize Σ by

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + (16 - r^2\cos^2\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k} \text{ for } 0 \le \theta \le 2\pi \text{ and } 0 \le r \le 2$$

Then

$$
\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\mathbf{i} - 2r\cos^2\theta\,\mathbf{j} + \sin\theta\,\mathbf{k}\\
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + 2r^2\sin\theta\cos\theta\,\mathbf{j} + r\cos\theta\,\mathbf{k}\\
\mathbf{r}_r \times \mathbf{r}_\theta &= -2r^2\cos\theta\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k}
\end{aligned}
$$

This is opposite Σ's and so:

$$\int\int_\Sigma (z\,\mathbf{i} + x^2\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n}\,dS = -\int\int_R (r\sin\theta\,\mathbf{i} + (r\cos\theta)^2\,\mathbf{j} + 1\,\mathbf{k}) \cdot (-2r^2\cos\theta\,\mathbf{i} - r\,\mathbf{j} + 0\,\mathbf{k})\,dA$$
$$= -\int_0^{2\pi}\int_0^2 -2r^3\sin\theta\cos\theta - r^3\cos^2\theta\,dr\,d\theta$$

10. Parts (a) and (b) are independent.

(a) Evaluate $\int_C (2x+y)\,dx + x\,dy$ where C is the line segment joining $(1,4)$ to $(5,-1)$. [10 pts]

Solution:

The vector field is conservative with $f(x,y) = x^2 + xy$ and so

$$\int_C (2x+y)\,dx + x\,dy = f(5,-1) - f(1,4) = \left[(5)^2 + (5)(-1)\right] - \left[(1)^2 + (1)(4)\right]$$

(b) Evaluate $\int\int_\Sigma (2x\,\mathbf{i} + 5z\,\mathbf{j} - 7y\,\mathbf{k}) \cdot \mathbf{n}\,dS$ where Σ is the sphere $x^2+y^2+z^2=9$ with inwards orientation. [10 pts]

Solution:

The surface Σ is the boundary of D where D is the solid ball $x^2+y^2+z^2 \le 9$. By the Divergence Theorem with a negative due to orientation we have:

$$\int\int_\Sigma (2x\,\mathbf{i} + 5z\,\mathbf{j} - 7y\,\mathbf{k}) \cdot \mathbf{n}\,dS = -\int\int\int_D 2 + 0 - 7\,dV$$
$$= 5(\text{Volume of } D)$$
$$= 5\left(\frac{4}{3}\pi(3)^3\right)$$