1. Parts (a) and (b) are independent.

(a) Let \( P = (-1, 3) \), \( Q = (-2, 1) \), and \( R = (1, -4) \). Find the projection of \( \overrightarrow{PQ} \) onto \( \overrightarrow{QR} \). [10 pts]

**Solution:**
We have \( \overrightarrow{PQ} = -1 \mathbf{i} - 2 \mathbf{j} \) and \( \overrightarrow{QR} = 3 \mathbf{i} - 5 \mathbf{j} \) and so
\[
\text{Proj}_{\overrightarrow{QR}} \overrightarrow{PQ} = \frac{(-1 \mathbf{i} - 2 \mathbf{j}) \cdot (3 \mathbf{i} - 5 \mathbf{j})}{(3 \mathbf{i} - 5 \mathbf{j}) \cdot (3 \mathbf{i} - 5 \mathbf{j})} (3 \mathbf{i} - 5 \mathbf{j}) = \frac{7}{34} (3 \mathbf{i} - 5 \mathbf{j})
\]

(b) Two forces \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) are applied to an object located at the origin in the xy-plane. The force \( \mathbf{F}_1 \) has a magnitude of 60 and makes an angle of \( \frac{\pi}{6} \) with the positive x-axis and force \( \mathbf{F}_2 \) has a magnitude of 100 and makes an angle of \( \frac{3\pi}{4} \) with the positive x-axis. Find the magnitude and direction (angle from positive x-axis) of the sum of these forces.

**Solution:**
The forces are
\[
\mathbf{F}_1 = 60 \cos \frac{\pi}{6} \mathbf{i} + 60 \sin \frac{\pi}{6} \mathbf{j} = 30\sqrt{3} \mathbf{i} + 30 \mathbf{j}
\]
and
\[
\mathbf{F}_2 = 100 \cos \frac{3\pi}{4} \mathbf{i} + 100 \sin \frac{3\pi}{4} \mathbf{j} = -50\sqrt{2} \mathbf{i} + 50\sqrt{2} \mathbf{j}
\]
and so
\[
\mathbf{F}_1 + \mathbf{F}_2 = (30\sqrt{3} - 50\sqrt{2}) \mathbf{i} + (30 + 30\sqrt{2}) \mathbf{j}
\]
The magnitude is then:
\[
\sqrt{(30\sqrt{3} - 50\sqrt{2})^2 + (30 + 30\sqrt{2})^2}
\]
and the angle with the positive x-axis is:
\[
\tan^{-1} \left( \frac{30 + 30\sqrt{2}}{30\sqrt{3} - 50\sqrt{2}} \right)
\]

2. Parts (a) and (b) are independent.

(a) Find the distance from the point \((1, -3, 2)\) to the line containing the points \((1, 0, 1)\) and \((5, 2, 0)\). [10 pts]

**Solution:** The line has \( \mathbf{L} = 4 \mathbf{i} + 2 \mathbf{j} - 1 \mathbf{k} \). If \( P = (1, 0, 1) \) is on the line and \( Q = (1, -3, 2) \) is off the line then the distance is:
\[
\frac{||\overrightarrow{PQ} \times \mathbf{L}||}{||\mathbf{L}||} = \frac{||(0 \mathbf{i} - 3 \mathbf{j} + 1 \mathbf{k}) \times (4 \mathbf{i} + 2 \mathbf{j} - 1 \mathbf{k})||}{||4 \mathbf{i} + 2 \mathbf{j} - 1 \mathbf{k}||}
\]
\[
= \frac{||1 \mathbf{i} + 4 \mathbf{j} + 12 \mathbf{k}||}{\sqrt{16 + 4 + 1}}
\]
\[
= \frac{\sqrt{1 + 16 + 144}}{\sqrt{16 + 4 + 1}}
\]
(b) Find an equation of the form \( ax + by + cz = d \) of the plane that contains the point \((1, 2, 3)\) and the line \[ \frac{x}{2} = \frac{y + 3}{7} = \frac{z - 4}{5}. \]

**Solution:**
The line has direction vector \( \mathbf{L} = 2 \mathbf{i} + 7 \mathbf{j} + 5 \mathbf{k} \) parallel to the plane and the vector \( \mathbf{a} = -1 \mathbf{i} - 5 \mathbf{j} + 1 \mathbf{k} \) joining \((1, 2, 3)\) to \((0, -3, 4)\) is also parallel to the plane. Thus:

\[
\mathbf{N} = (2 \mathbf{i} + 7 \mathbf{j} + 5 \mathbf{k}) \times (-1 \mathbf{i} - 5 \mathbf{j} + 1 \mathbf{k}) = 32 \mathbf{i} - 7 \mathbf{j} - 3 \mathbf{k}
\]

and so the plane is

\[
32(x - 1) - 7(y - 2) - 3(z - 3) = 0
\]

3. Parts (a) and (b) are independent.

(a) Find a parametrization of the quarter circle in the plane \( z = 1 \) with endpoints \((-1, 0, 1)\) and \((0, -1, 1)\) and center \((0, 0, 1)\).

**Solution:**
The most obvious answer might be:

\[
\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 1 \mathbf{k} \quad \text{for} \quad \pi \leq t \leq \frac{3\pi}{2}
\]

(b) Determine if the following pair of lines is parallel, intersecting, or neither. If the lines intersect, find the point at which they intersect.

\[
\mathbf{r}_1(t) = (1 + 6t)\mathbf{i} + (3 - 7t)\mathbf{j} + (2 + t)\mathbf{k}
\]

\[
\mathbf{r}_2(s) = (10 + 3s)\mathbf{i} + (6 + s)\mathbf{j} + (14 + 4s)\mathbf{k}
\]

**Solution:**
In order to meet we would need:

\[
1 + 6t = 10 + 3s
\]
\[
3 - 7t = 6 + s
\]
\[
2 + t = 14 + 4s
\]

The second states that \( s = -7t - 3 \) so if we plug into the third we get \( 2 + t = 14 + 4(-7t - 3) \) which yields \( t = 0 \) and so \( s = -3 \). These satisfy the first so they meet.

Since they meet at \( t = 0 \) this is the point \( \mathbf{r}_1(0) = 1 \mathbf{i} + 3 \mathbf{j} + 2 \mathbf{k} \) or \((1, 3, 2)\).
4. Find all critical points of \( f(x, y) = x^3 - 6x^2 - 5y^2 \) and classify each as a relative minimum, maximum, or saddle point.

**Solution:**
The critical points occur when
\[
 f_x = 3x^2 - 12x = 3x(x - 4) = 0
\]
\[
 f_y = -10y = 0
\]
Thus \( y = 0 \) and \( x = 0, 4 \) yielding \((0, 0)\) and \((4, 0)\).

We have \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (6x - 12)(-10) - (0)^2 \) and so:
\[
 D(0, 0) = + \text{ and } f_{yy}(0, 0) = - \text{ so relative maximum.}
 \]
\[
 D(4, 0) = - \text{ so saddle point.}
 \]

5. Use Lagrange Multipliers to find the maximum and minimum of \( f(x, y) = x^3 - 6x^2 - 5y^2 \) subject to the constraint \( x^2 + y^2 = 1 \).

**Solution:**
We solve the system:
\[
 3x^2 - 12x = \lambda 2x
\]
\[
 -10y = \lambda 2y
\]
\[
 x^2 + y^2 = 1
\]

The first tells us that \( x = 0 \) or \( \lambda = \frac{3x-12}{2} \). If \( x = 0 \) then the third tells us \( y = \pm 1 \) so \((0, \pm 1)\).

The second tells us that \( y = 0 \) or \( \lambda = -5 \). If \( y = 0 \) then the third tells us \( x = \pm 1 \) so \((\pm 1, 0)\).

Otherwise \( \frac{3x-12}{2} = -5 \) and so \( 3x = 2 \) so \( x = \frac{2}{3} \) so the third tells us that \( y = \pm \sqrt{5/9} \) so \((2/3, \pm \sqrt{5/9})\).

Then:
- \( f(0, +1) = -5 \)
- \( f(0, -1) = -5 \)
- \( f(+1, 0) = -5 \)
- \( f(-1, 0) = -7 \)
- \( f(2/3, +\sqrt{5/9}) = \frac{8}{27} - 6 \left( \frac{4}{3} \right) - 5 \left( \frac{5}{9} \right) = -\frac{139}{27} \)
- \( f(2/3, -\sqrt{5/9}) = \frac{8}{27} - 6 \left( \frac{4}{3} \right) - 5 \left( \frac{5}{9} \right) = -\frac{139}{27} \)

Noting \(-5 = -\frac{139}{27}\) and \(-7 = -\frac{139}{27}\) we see that the maximum is \(-5\) and the minimum is \(-7\).

6. Let \( R \) be the region in the first quadrant of the \( xy \)-plane bounded by the lines \( y = x \) and \( y = 3x \) and by the hyperbolas \( xy = 1 \) and \( xy = 5 \). Use the change of variables \( x = \frac{u}{v} \) and \( y = v \) to set
up an iterated double integral in the \(uv\)-plane representing the area of \(R\).

**Do not evaluate this integral.**

**Solution:**

We have the new region \(S\) defined by:

\[
y = x \rightarrow v = \frac{u}{v} \rightarrow v = \sqrt{u}
\]
\[
y = 3x \rightarrow v = \frac{3u}{v} \rightarrow v = \sqrt{3u}
\]
\[
xy = 1 \rightarrow u = 1
\]
\[
xy = 5 \rightarrow u = 5
\]

We have:

\[
J = \det \begin{bmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{bmatrix} = 1/v
\]

Therefore:

\[
\text{Area} = \int \int_{R} 1 \, dA = \int \int_{S} |1/v| \, dA = \int_{1}^{\sqrt{3}} \int_{1}^{\sqrt{3u}} 1/v \, dv \, du
\]

7. Parts (a) and (b) are independent.

(a) Find the unit vector direction of maximum increase of \(f(x, y) = x^2 y - xy\) at the point \((2, -3)\).  

**Solution:**

The direction of maximum increase is:

\[
\nabla f(x, y) = (2xy - y) \, \hat{i} + (x^2 - x) \, \hat{j}
\]
\[
\nabla f(2, -3) = (2(2)(-3) - (-3)) \, \hat{i} + ((2)^2 - 2) \, \hat{j}
\]
\[
= -9 \, \hat{i} + 2 \, \hat{j}
\]

Therefore the corresponding unit vector is:

\[
\frac{-9 \, \hat{i} + 2 \, \hat{j}}{\sqrt{(-9)^2 + 4^2}}
\]

(b) Use tangent plane approximation to approximate \(f(16.05)^{1/4}(7.95)^{2/3}\).  

**Solution:**

We set \(f(x, y) = x^{1/4} y^{2/3}\) and we wish to approximate \(f(16.05, 7.95)\). We use the anchor point \((16, 8)\) and then since:
\[ f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ f(x, y) \approx x_0^{1/4}y_0^{2/3} + \frac{1}{4}x_0^{-3/4}y_0^{2/3}(x - x_0) + \frac{2}{3}x_0^{1/4}y_0^{-1/3}(y - y_0) \]

\[ f(16.05, 7.95) \approx 16^{1/4}8^{2/3} + \frac{1}{4}(16)^{-3/4}(8)^{2/3}(16 - 16.05) + \frac{2}{3}(16)^{1/4}(8)^{-1/3}(8 - 7.95) \]

8. Write down the iterated triple integral representing the volume of the solid region \( D \) bounded between the spheres (centered at the origin) of radius 1 and 3, and the upper part of the cone \( z^2 = 3(x^2 + y^2) \).

**Do not evaluate this integral.**

**Solution:**

We have:

\[ \int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

9. Let \( C \) be the intersection of the parabolic sheet \( y = 16 - x^2 \) with the cylinder \( x^2 + z^2 = 4 \) [25 pts] with counterclockwise orientation when viewed from the positive \( y \)-axis. Use Stokes’ Theorem to convert the line integral

\[ \int_C x^2 z \, dx + x \, dy + yz \, dz \]

to a surface integral. Parametrize the surface to obtain a double iterated integral.

**Do not evaluate this integral.**

**Solution:**

By Stokes’ Theorem we have:

\[ \int_C x^2 z \, dx + x \, dy + yz \, dz = \int \int_{\Sigma} (z \, i + x^2 \, j + 1 \, k) \cdot n \, dS \]

where \( \Sigma \) is the portion of the parabolic sheet inside the cylinder, oriented in the positive \( y \)-direction.

We parametrize \( \Sigma \) by

\[ r(r, \theta) = r \cos \theta \, i + (16 - r^2 \cos^2 \theta) \, j + r \sin \theta \, k \]

for \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 2 \)

Then

\[ r_r = \cos \theta \, i - 2r \cos^2 \theta \, j + \sin \theta \, k \]

\[ r_\theta = -r \sin \theta \, i + 2r^2 \sin \theta \cos \theta \, j + r \cos \theta \, k \]

\[ r_r \times r_\theta = -2r^2 \cos \theta \, i - r \, j + 0 \, k \]

This is opposite \( \Sigma \)'s and so:
\[ \int \int_{\Sigma} \left( z \mathbf{i} + x^2 \mathbf{j} + 1 \mathbf{k} \right) \cdot \mathbf{n} \, dS = - \int \int_{R} \left( r \sin \theta \mathbf{i} + (r \cos \theta)^2 \mathbf{j} + 1 \mathbf{k} \right) \cdot \left( -2r^2 \cos \theta \mathbf{i} - r \mathbf{j} + 0 \mathbf{k} \right) \, dA \\
= - \int_0^{2\pi} \int_0^2 -2r^3 \sin \theta \cos \theta - r^3 \cos^2 \theta \, dr \, d\theta \]

10. Parts (a) and (b) are independent.

(a) Evaluate \( \int_C (2x + y) \, dx + x \, dy \) where \( C \) is the line segment joining \((1, 4)\) to \((5, -1)\). [10 pts]

**Solution:**

The vector field is conservative with \( f(x, y) = x^2 + xy \) and so

\[ \int_C (2x + y) \, dx + x \, dy = f(5, -1) - f(1, 4) = [(5)^2 + (5)(-1)] - [(1)^2 + (1)(4)] \]

(b) Evaluate \( \int \int_{\Sigma} (2x \, \mathbf{i} + 5z \, \mathbf{j} - 7y \, \mathbf{k}) \cdot \mathbf{n} \, dS \) where \( \Sigma \) is the sphere \( x^2 + y^2 + z^2 = 9 \) with inwards orientation. [10 pts]

**Solution:**

The surface \( \Sigma \) is the boundary of \( D \) where \( D \) is the solid ball \( x^2 + y^2 + z^2 \leq 9 \). By the Divergence Theorem with a negative due to orientation we have:

\[ \int \int_{\Sigma} (2x \, \mathbf{i} + 5z \, \mathbf{j} - 7y \, \mathbf{k}) \cdot \mathbf{n} \, dS = - \int \int \int_D 2 + 0 - 7 \, dV \]
\[ = 5(\text{Volume of } D) \]
\[ = 5 \left( \frac{4}{3} \pi (3)^3 \right) \]