1. Parts (a) and (b) are independent.

   (a) Find parametric equations for the line L containing the points $(-2, 0, 1)$ and $(4, -2, -3)$. [10 pts]

   Solution:

   Since:

$$\mathbf{L} = 6\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$$

   we have

$$\begin{aligned} x &= -2 + 6t \\ y &= 0 - 2t \\ z &= 1 - 4t \end{aligned}$$

   (b) Do the planes $\mathcal{P}_0 : 2x - y + 3z = -2$, and $\mathcal{P}_1 : -2x - 3y + z = 6$ intersect? If so, find symmetric equations for the line of intersection. If not, explain why not. [10 pts]

   Solution:

   Yes since the normal vectors $\mathbf{n}_1 = 2\,\mathbf{i} - 1\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{n}_2 = -2\,\mathbf{i} - 3\,\mathbf{j} + 1\,\mathbf{k}$ are not parallel.
   For the line between them we have:

$$\mathbf{L} = \mathbf{n}_1 \times \mathbf{n}_2 = 8\,\mathbf{i} - 8\,\mathbf{j} - 8\,\mathbf{k}$$

   If we add the equations we get $-4y + 4z = 4$ so we may use $z = 1$ and $y = 0$ and then $x = -\frac{5}{2}$.
   The solution is therefore:

$$\frac{x - \left(-\frac{5}{2}\right)}{8} = \frac{y - 0}{-8} = \frac{z - 1}{-8}$$

2. (a) Find an equation for the plane containing the points $P = (1, -3, 1)$, $Q = (2, 2, 0)$, and $R = (-4, -1, 1)$. [10 pts]

Solution:

We have:

$$\begin{aligned}
\vec{PQ} &= 1\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k} \\
\vec{PR} &= -5\,\mathbf{i} + 2\,\mathbf{j} + 0\,\mathbf{k} \\
\vec{PQ} \times \vec{PR} &= 2\,\mathbf{i} + 5\,\mathbf{j} + 27\,\mathbf{k}
\end{aligned}$$

So the plane is:

$$2(x-1) + 5(y+3) + 27(z-1) = 0$$

Note: They don't need to simplify but for comparison this simplifies to:

$$2x + 5y + 27z = 14$$

(b) Find the distance between the point $S = (-2, 3, 1)$ and the plane $-4x + y - 2z = 0$. [10 pts]

Solution:

The normal vector for the plane is $\mathbf{n} = -4\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k}$ and a point on the plane is $P = (0, 0, 0)$ so then we have $\vec{PS} = -2\,\mathbf{i} + 3\,\mathbf{j} + 1\,\mathbf{k}$ and so:

$$\text{dist} = \frac{|\vec{PS} \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|(-2\,\mathbf{i} + 3\,\mathbf{j} + 1\,\mathbf{k}) \cdot (-4\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k})|}{||-4\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k}||} = \frac{9}{\sqrt{21}}$$

3. Parts (a) and (b) are independent.

(a) Compute the length of the curve C_1 with parametrization [10 pts]

$$\mathbf{r}(t) = \tfrac{1}{3}(1+t)^{\frac{3}{2}}\,\mathbf{i} + \tfrac{1}{3}(1-t)^{\frac{3}{2}}\,\mathbf{j} + t\sqrt{3}\,\mathbf{k} \text{ for } -\tfrac{1}{4} \le t \le \tfrac{1}{2}$$

Solution:

We have:

$$\begin{aligned}
\mathbf{r}'(t) &= \frac{1}{2}(1+t)^{\frac{1}{2}}\,\mathbf{i} - \frac{1}{2}(1-t)^{\frac{1}{2}}\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \\
||\mathbf{r}'(t)|| &= \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + 3} \\
&= \sqrt{\frac{7}{2}}
\end{aligned}$$

and so:

$$\text{Length} = \int_{-\frac{1}{4}}^{\frac{1}{2}} \sqrt{\frac{7}{2}}\,dt = \sqrt{\frac{7}{2}}\left(\frac{1}{2}\right) - \sqrt{\frac{7}{2}}\left(-\frac{1}{4}\right)$$

(b) Find all points (if any) where the curve C_2 with the following parametrization meets the sphere of radius 3 centered at the origin. [10 pts]

$$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t+1}\,\mathbf{j} + t\,\mathbf{k} \text{ for } t \ge 0$$

Solution:

The sphere has equation $x^2 + y^2 + z^2 = 9$ and so we must have:

$$\begin{aligned}
\left(\sqrt{t}\right)^2 + \left(\sqrt{t+1}\right)^2 + t^2 &= 9 \\
t + t + 1 + t^2 &= 9 \\
t^2 + 2t - 8 &= 0 \\
(t+4)(t-2) &= 0
\end{aligned}$$

Since $t \ge 0$ we have $t = 2$ and the point is:

$$\mathbf{r}(4) = \sqrt{2}\,\mathbf{i} + \sqrt{3}\,\mathbf{j} + 2\,\mathbf{k}$$

which is the point:

$$\left(\sqrt{2}, \sqrt{3}, 2\right)$$

4. Consider the curve parameterized by $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$.

(a) Find the tangent vector $\mathbf{T}(t)$. [12 pts]

Solution:

We have:

$$\mathbf{r}'(t) = -3\sin t\cos^2 t\,\mathbf{i} + 3\sin^2 t\cos t\,\mathbf{j}$$

Noting that:

$$\begin{aligned}||\mathbf{r}'(t)|| &= \sqrt{9\sin^2\cos^4 t + 9\sin^4\cos^2 t} \\ &= 3\sin t\cos t\sqrt{\cos^2 t + \sin^2 t} \\ &= 3\sin t\cos t\end{aligned}$$

We then have:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$$

(b) Find the normal vector $\mathbf{N}(t)$. [8 pts]

Solution:

We have:

$$\mathbf{T}'(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

noting that:

$$||\mathbf{T}'(t)|| = 1$$

we then have:

$$\mathbf{N}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

5. Let $f(x, y, z) = z^2 - 8\sqrt{x^2 - 3y}$. Consider the S surface with equation $f(x, y, z) = 0$.

(a) Find grad$f$. [5 pts]

Solution:

We have:

$$\nabla f = -4(x^2 - 3y)^{-\frac{1}{2}}(2x)\,\mathbf{i} - 4(x^2 - 3y)^{-\frac{1}{2}}(-3)\,\mathbf{j} + 2z\,\mathbf{k}$$

(b) Find an equation of the plane tangent to the level surface for f at $(5, 3, 2)$. [5 pts]

Solution:

Since:

$$\begin{aligned}\nabla f(5,3,2) &= -4(25-9)^{-\frac{1}{2}}(10)\,\mathbf{i} - 4(25-9)^{-\frac{1}{2}}(-3)\,\mathbf{j} + 4\,\mathbf{k} \\ &= -10\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}\end{aligned}$$

So the plane has equation:

$$-10(x-5) + 3(y-3) + 4(z-2) = 0$$

(c) Find $D_{\mathbf{u}}f$ at $(5, 3, 2)$ where $\mathbf{u}$ is pointing in the direction $1\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$. [5 pts]

Solution:

We have:

$$\mathbf{u} = \frac{1}{\sqrt{14}}(1\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k})$$

and so:

$$\begin{aligned}D_{\mathbf{u}}f(5,3,2) &= \frac{1}{\sqrt{14}}(1\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}) \cdot (-10\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}) \\ &= \frac{1}{\sqrt{14}}(-10 + 6 - 12)\end{aligned}$$

(d) Find the smallest value of $D_{\mathbf{u}}f$ at $(5, 3, 2)$. [5 pts]

Solution:

This would be:

$$-||\nabla f(5,3,2)|| = -\sqrt{100 + 9 + 16}$$

6. Let $f(x,y) = x^4 + y^2$.

(a) Use Lagrange Multipliers to find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2 + y^2 = 1$. [15 pts]

Solution:

We set $g(x,y) = x^2 + y^2$ and solve the system:

$$\begin{aligned} 4x^3 &= \lambda 2x \\ 2y &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned}$$

Note that $x = 0$ satisfies the first and yields $y = \pm 1$ in the third, and this also satisfies the second. Thus we have $(0, \pm 1)$.

Note that $y = 0$ satisfies the second and yields $x = \pm 1$ in the third, and this also satisfies the first. Thus we have $(\pm 1, 0)$.

If neither is 0 then solving the first and second for λ and equating yields $2x^2 = 1$ or $x = \pm\frac{\sqrt{2}}{2}$ which in the third yields $\pm\frac{\sqrt{2}}{2}$.

Thus we have eight points which we check: Checking:

$$f(1,0) = f(-1,0) = f(0,1) = f(0,-1) = 1$$

and:

$$f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{3}{4}$$

So the maximum is 1 and the minimum is $\frac{3}{4}$.

(b) Find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2 + y^2 \leq 1$. [5 pts]

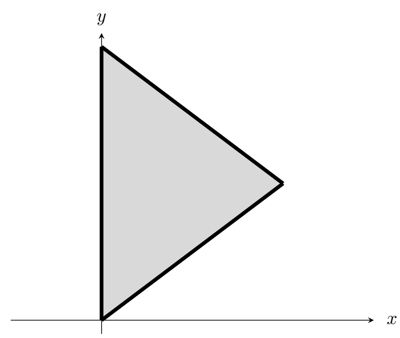
Solution: We have $f_x = 2x$ and $f_y = 2y$ which yields critical point $(0,0)$. Noting that $f(0,0) = 0$ and that the maximum and minimum on the boundary were determined we still have a maximum of 1 and now a minimum of 0.

7. Use the change of variables $u = y - x$ and $v = y + x$ to evaluate the double integral [20 pts]

$$\iint_R (y-x)\sin\left((y+x)^3\right)\, dA$$

where R is the triangle with vertices $(0,0)$, $(2,2)$ and $(0,4)$.

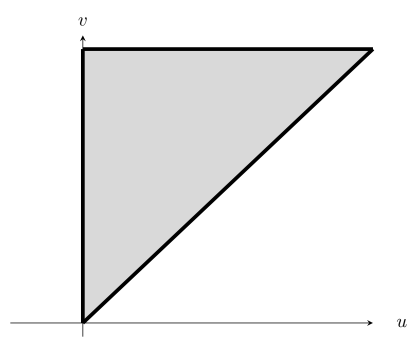
Solution: The region looks like:



The sides of the triangle are $y = x$, $y = 4 - x$ and $x = 0$.

- Rewriting the first two as $y - x = 0$, $y + x = 4$ yields $u = 0$ and $v = 4$.
- The COV yields $x = \frac{1}{2}(v-u)$ and $y = \frac{1}{2}(u+v)$ so $x = 0$ yields $\frac{1}{2}(v-u) = 0$ or $v = u$.

Thus the new region S is:



The Jacobian of the change of variables is:

$$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Thus we have:

$$\begin{aligned}
\iint_R (y-x)\sin\left((y+x)^3\right)\, dA &= \iint_S u\sin(v^3)\left|-\frac{1}{2}\right|\, dA \\
&= \frac{1}{2}\int_0^4\int_0^v u\sin(v^3)\, du\, dv \\
&= \frac{1}{4}\int_0^4 u^2\sin(v^3)\Big|_0^v\, dv \\
&= \frac{1}{4}\int_0^4 v^2\sin(v^3)\, dv \\
&= -\frac{1}{12}\cos(v^3)\Big|_0^4 \\
&= -\frac{1}{12}(\cos(64) - \cos(0))
\end{aligned}$$

8. Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the paraboloid $2z = x^2 + y^2$. [20 pts]

Solution:

The solid is the region above the paraboloid and below the sphere. In cylindrical the sphere is $r^2 + z^2 = 8$ or (the top half is) $z = \sqrt{8 - r^2}$ and the paraboloid is $2z = r^2$ or $z = \frac{1}{2}r^2$.

The two meet when:

$$\begin{aligned} r^2 + z^2 &= 8 \\ 2z + z^2 &= 8 \\ z^2 + 2z - 8 &= 0 \\ (z-2)(z+4) &= 0 \end{aligned}$$

Since $z \geq 0$ we have $z = 2$ and so $r^2 = 4$ so $r = 2$.

The volume therefore is:

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^2 \left[\sqrt{8 - r^2} - \frac{1}{2}r^2\right] r\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^2 r\sqrt{8 - r^2} - \frac{1}{2}r^3\,dr\,d\theta \\ &= \int_0^{2\pi} -\frac{1}{3}(8 - r^2)^{\frac{3}{2}} - \frac{1}{8}r^4 \Big|_0^2\,d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{\frac{3}{2}} - \frac{1}{8}(16)\right] - \left[-\frac{1}{3}(8)^{\frac{3}{2}} - \frac{1}{8}(0)\right]\,d\theta \\ &= 2\pi\left[\left[-\frac{1}{3}(4)^{\frac{3}{2}} - \frac{1}{8}(16)\right] - \left[-\frac{1}{3}(8)^{\frac{3}{2}} - \frac{1}{8}(0)\right]\right] \end{aligned}$$

9. Parts (a) and (b) are independent.

(a) Let Σ be the portion of the cylinder $x^2 + y^2 = 9$ between $z = 1$ and $z = 8$. If the mass density at (x, y, z) is given by $f(x, y, z) = x^2 z$ write down an iterated double integral for the mass of Σ. [12 pts]

Do not evaluate the integral!

Solution:

We parametrize the cylinder by:

$$\mathbf{r}(\theta, z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k} \text{ with } 0 \leq \theta \leq 2\pi \text{ and } 1 \leq z \leq 8 \text{ (this is } R\text{)}$$

Therefore we have

$$\begin{aligned}
\mathbf{r}_\theta &= -3\sin\theta\,\mathbf{i} + 3\cos\theta\,\mathbf{j} + 0\,\mathbf{k} \\
\mathbf{r}_z &= 0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{r}_\theta \times \mathbf{r}_z &= 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + 0\,\mathbf{k} \\
||\mathbf{r}_\theta \times \mathbf{r}_z|| &= 3
\end{aligned}$$

And so:

$$\begin{aligned}
\text{Mass} &= \iint_\Sigma x^2 z\, dS \\
&= \iint_R (3\cos\theta)^2 (z)(3)\, dA \\
&= \int_0^{2\pi} \int_1^8 27z\cos^2\theta\, dz\, d\theta
\end{aligned}$$

(b) Let C be the triangle with vertices $(0, 4)$, $(2, 0)$ and $(2, 4)$ with clockwise orientation. Use Green's Theorem to evaluate [8 pts]

$$\int_C 4y\, dx + 9x\, dy$$

Solution:

We have:

$$\int_C 4y\, dx + 9x\, dy = \iint_R 9 - 4\, dA = 5 \iint_R 1\, dA = 5(\text{Area of } R) = 5\left(\frac{1}{2}(2)(4)\right)$$

10. Let Σ be the portion of the plane $x + 2y + z = 10$ in the first octant. Let C be the boundary of Σ with counterclockwise orientation when viewed from above. Use Stokes' Theorem to rewrite the integral $\int_C 3xy\,dx + z^2\,dy + xy\,dz$ as a surface integral and then proceed until you have an iterated double integral. **Do not evaluate the integral!** [20 pts]

Solution:

Stokes' Theorem tells us that

$$\int_C 3xy\,dx + z^2\,dy + xy\,dz = \iint_\Sigma [(x-2z)\,\mathbf{i} - (y-0)\,\mathbf{j} + (0-3x)\,\mathbf{k}] \cdot \mathbf{n}\,dS$$

where Σ is the part of the plane in the first octant oriented upwards. We parametrize Σ as:

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (10 - x - 2y)\,\mathbf{k} \text{ with } 0 \le x \le 10 \text{ and } 0 \le y \le 5 - \tfrac{1}{2}x \text{ (this is } R)$$

Then

$$\begin{aligned} \mathbf{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k} \\ \mathbf{r}_x \times \mathbf{r}_y &= 1\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k} \end{aligned}$$

Since this matches Σ's orientation the integral becomes

$$\iint_\Sigma [(x-2z)\,\mathbf{i} - (y-0)\,\mathbf{j} + (0-3x)\,\mathbf{k}] \cdot \mathbf{n}\,dS$$

$$\begin{aligned} &= +\iint_R [(x - 2(10-x-2y))\,\mathbf{i} - y\,\mathbf{j} - 3x\,\mathbf{k}] \cdot [1\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}]\ dA \\ &= \int_0^{10}\int_0^{5-\frac{1}{2}x} x - 2(10-x-2y) - 2y - 3x\,dy\,dx \end{aligned}$$