Math 241 Final Spring 2020 Solution **Justin Wyss-Gallifent**

1. Let $\mathcal{L}$ be the line with symmetric equation:

$$\frac{x-1}{2} = \frac{z}{3}\ ,\ y = 2$$

Let $\mathcal{P}$ be the plane with equation:

$$2x + y - 3z = 1$$

(a) Find the point at which $\mathcal{L}$ meets $\mathcal{P}$. [10 pts]

Solution: The line has $z = \frac{3x-3}{2}$ and $y = 2$ so we can rewrite the plane:

$$\begin{aligned}
2x + 2 - 3\left(\frac{3x-3}{2}\right) &= 1 \\
2x + 2 - \frac{9x-9}{2} &= 1 \\
4x + 4 - 9x + 9 &= 2 \\
-5x &= -11 \\
x &= \frac{11}{5}
\end{aligned}$$

Then:

$$z = \frac{3x-3}{2} = \frac{3(11/5)-3}{2} = \frac{33/5-3}{2} = \frac{9}{5}$$

Thus the point is:

$$\left(\frac{11}{5}, 2, \frac{9}{5}\right)$$

(b) At this point, what is the cosine of the angle between $\mathcal{L}$ and the normal vector for $\mathcal{P}$? [15 pts]

Solution: We know that the vector for $\mathcal{L}$ is $\mathbf{L} = 2\,\mathbf{i} + 0\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{N} = 2\,\mathbf{i} + 1\,\mathbf{j} - 3\,\mathbf{k}$. Then:

$$\cos\theta = \frac{\mathbf{L}\cdot\mathbf{N}}{||\mathbf{L}||\,||\mathbf{N}||} = \frac{-5}{\sqrt{13}\sqrt{14}}$$

(c) If you went to the point on the line where $x = 7$ and drew a sphere centered at that point, how large could the radius be before the sphere hit the plane? [15 pts]

Solution: The point with $x = 7$ has:

$$\frac{7-1}{2} = \frac{z}{3}\ ,\ y = 2$$

Hence is $(7, 2, 9)$. We then find the distance from $Q = (7, 2, 9)$ to the plane. The plane has $P = (0, 1, 0)$ and so the distance is:

$$\frac{|\overrightarrow{PQ}\cdot\mathbf{N}|}{||\mathbf{N}||} = \frac{|(7\,\mathbf{i} + 1\,\mathbf{j} + 9\,\mathbf{k})\cdot(2\,\mathbf{i} + 1\,\mathbf{j} - 3\,\mathbf{k})|}{||2\,\mathbf{i} + 1\,\mathbf{j} - 3\,\mathbf{k}||} = \frac{|-12|}{\sqrt{14}}$$

So the radius must be smaller than this.

2. Define the function:

$$f(x, y) = xy^2 + 2xy - y$$

(a) Find the equation of the plane tangent to the graph of $f(x, y)$ at $(1, 2)$. Write this in the form $ax + by + cz = d$. [25 pts]

Solution: The graph is $z = xy^2 + 2xy - y$. At $(x, y) = (1, 2)$ we have $z = 6$. Then if we introduce a new $g(x, y, z)$ as follows:

$$\begin{aligned} g(x, y, z) &= xy^2 + 2xy - y - z \\ \nabla g(x, y, z) &= (y^2 + 2y)\,\mathbf{i} + (2xy + 2x - 1)\,\mathbf{j} - 1\,\mathbf{k} \\ \nabla g(1, 2, 6) &= 8\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k} \end{aligned}$$

So the plane is:

$$\begin{aligned} 8(x - 1) + 5(y - 2) - 1(z - 6) &= 0 \\ 8x + 5y - z &= 12 \end{aligned}$$

(b) Calculate $\int_C \nabla f(x, y) \cdot d\mathbf{r}$ where C is the curve parametrized by $\mathbf{r}(t) = (t^2 + t)\,\mathbf{i} + (5t + 2)\,\mathbf{j}$ for $1 \leq t \leq 2$. [15 pts]

Solution: Since $\nabla f(x, y)$ is conservative with potential function $f(x, y)$ we can use the Fundamental Theorem of Line Integrals. The curve has:

$$\begin{aligned} \text{Start} \quad & \mathbf{r}(1) = 2\,\mathbf{i} + 7\,\mathbf{j} \implies (2, 7) \\ \text{End} \quad & \mathbf{r}(2) = 6\,\mathbf{i} + 12\,\mathbf{j} \implies (6, 12) \end{aligned}$$

Then:

$$\int_C \nabla f(x, y) \cdot d\mathbf{r} = f(6, 12) - f(2, 7) = \big((6)(12)^2 + 2(6)(12) - 12\big) - \big((2)(7)^2 + 2(2)(7) - 7\big)$$

3. Let C be the counterclockwise curve consisting of the semicircle $x^2 + y^2 = 9$ with $x \geq 0$ along with the line segment joining the endpoints of that semicircle. Consider the integral:

$$\int_C 6y\,dx + 15x\,dy$$

(a) Parametrize C (you'll need two parametrizations) and use them to evaluate the integral. [20 pts]

Solution: For the curved part of C we have $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and then:

$$\begin{aligned}
\int_C 6y\,dx + 15x\,dy &= \int_{-\pi/2}^{\pi/2} 6(3\sin t)(-3\sin t) + 15(3\cos t)(3\cos t)\,dt \\
&= \int_{-\pi/2}^{\pi/2} 6(3\sin t)(-3\sin t) + 15(3\cos t)(3\cos t)\,dt \\
&= \int_{-\pi/2}^{\pi/2} -54 + 189\cos^2 t\,dt \\
&= \int_{-\pi/2}^{\pi/2} -54 + \frac{189}{2}(1 + \cos(2t))\,dt \\
&= -54t + \frac{189}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{81\pi}{2}
\end{aligned}$$

For the straight part of C we have $\mathbf{r}(t) = 0\,\mathbf{i} + (3 - t)\,\mathbf{j}$ for $0 \leq t \leq 6$. and so $\mathbf{r}'(t) = 0\,\mathbf{i} - 1\,\mathbf{j}$ and then:

$$\int_C 6y\,dx + 15x\,dy = \int_0^6 6(3 - t)(0) + 15(0)(-1)\,dt = 0$$

Thus the total is $\frac{81\pi}{2}$.

(b) Use Green's Theorem to rewrite the line integral as an integral over a region R. Evaluate this integral. [15 pts]

Solution: Using Green's Theorem we have:

$$\begin{aligned}
\int_C 6y\,dx + 15x\,dy &= \iint_R 15 - 6\,dA \\
&= 9\iint_R 1\,dA \\
&= 9\left(\frac{\pi(3)^2}{2}\right) \\
&= \frac{81\pi}{2}
\end{aligned}$$

(c) Your answers to (a) and (b) should be equal. Are they? Yes or no is enough. [5 pts]

Solution: Yes.

4. Let C be the intersection of the plane $z = 9 - y$ with the cylinder $x^2 + y^2 = 4$ with counterclockwise orientation when viewed from above. Consider the integral:

$$\int_C x\,dx + x\,dy + z\,dz$$

(a) Parametrize C and use this parametrization to evaluate the integral. [15 pts]
Solution: We parametrize C using:

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + (9 - 2\sin t)\,\mathbf{k} \text{ with } 0 \le t \le 2\pi$$

Then we have:

$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 2\cos t\,\mathbf{k}$$

and then:

$$\begin{aligned}
\int_C x\,dx + x\,dy + z\,dz &= \int_0^{2\pi} (2\cos t)(-2\sin t) + (2\cos t)(2\cos t) + (9 - 2\sin t)(-2\cos t)\,dt \\
&= \int_0^{2\pi} -18\cos t + 4\cos^2 t\,dt \\
&= \int_0^{2\pi} -18\cos t + 2(1 + \cos(2t))\,dt \\
&= \int_0^{2\pi} -18\cos t + 2 + 2\cos(2t)\,dt \\
&= 18\sin t + 2t + \sin(2t)\Big|_0^{2\pi} \\
&= 4\pi
\end{aligned}$$

(b) Use Stokes' Theorem to rewrite the line integral as a surface integral over an appropriate surface. Evaluate this surface integral. [15 pts]
Solution: Stokes' Theorem states that:

$$\int_C x\,dx + x\,dy + z\,dz = \iint_\Sigma (0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n}\,dS$$

where Σ is the part of the plane inside the cylinder with induced upwards orientation. We then parametrize Σ and continue:

$$\begin{aligned}
\mathbf{r}(r,\theta) &= r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (9 - r\sin\theta)\,\mathbf{k} \qquad \text{with } 0 \le r \le 2, 0 \le \theta \le 2\pi \\
\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - \sin\theta\,\mathbf{k} \\
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} - r\cos\theta\,\mathbf{k} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= 0\,\mathbf{i} + r\,\mathbf{j} + r\,\mathbf{k}
\end{aligned}$$

This matches Σ's orientation and so we get:

$$\begin{aligned}
\iint_\Sigma (0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}) \cdot \mathbf{n}\,dS &= +\iint_R (0\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k}) \cdot (0\,\mathbf{i} + r\,\mathbf{j} + r\,\mathbf{k})\,dA \\
&= \int_0^{2\pi}\int_0^2 r\,dr\,d\theta \\
&= \int_0^{2\pi} 2\,d\theta \\
&= 4\pi
\end{aligned}$$

(c) Write down the Matlab command you would use to evaluate the final iterated integral in (b). [5 pts]
Solution: `int(int(r,0,2),0,2*pi)`

(d) Your answers to (a) and (b) should be equal. Are they? Yes or no is enough. [5 pts]
Solution: Yes.

5. Define the function:

$$f(x, y) = x^3 + y^3 + 3xy$$

(a) Find and categorize all critical points of $f(x, y)$. [20 pts]

Solution: We have:

$$f_x(x, y) = 3x^2 + 3y = 0$$
$$f_y(x, y) = 3y^2 + 3x = 0$$

The first gives us $y = -x^2$ and if we plug this into the second we get:

$$3(-x^2)^2 + 3x = 0$$
$$x^4 + x = 0$$
$$x(x^3 + 1) = 0$$

So we get $x = 0$ and $x = -1$.

If $x = 0$ then $y = 0$ and we have $(0, 0)$ and if $x = -1$ then $y = -1$ and we have $(-1, -1)$

We then find $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (3)^2$ and we check the points:

$D(0, 0) < 0$ so $(0, 0)$ is a saddle point.

$D(-1, -1) > 0$ and $f_{xx}(-1, -1) < 0$ so $(-1, -1)$ is a relative maximum.

(b) Without doing any integration explain how you know that $\int_C f(x, y)\, ds > 25$ where C is the part of the semicircle $x^2 + y^2 = 4$ in the first quadrant. [20 pts]

Hint: This is tricky. The integral measures mass. Can you find a constant A such that $f(x, y) \geq A$ on C?

Solution: If we set $g(x, y) = x^2 + y^2$ and use Lagrange multipliers to minimize f with $g(x, y) = 4$ we have the system:

$$3x^2 + 3y = \lambda(2x)$$
$$3y^2 + 3x = \lambda(2y)$$
$$x^2 + y^2 = 4$$

The first two can be rewritten:

$$3x^2y + 3y^2 = \lambda(2xy)$$
$$3xy^2 + 3x^2 = \lambda(2xy)$$

Then we have:

$$3x^2y + 3y^2 = 3xy^2 + 3x^2$$
$$x^2y - xy^2 = x^2 - y^2$$
$$xy(x - y) = (x - y)(x + y)$$

We then have either $x - y = 0$ or $xy = x + y$. The latter is a hyperbola which does not intersect $x^2+y^2 = 4$ so this yields no solutions. The former then gives us $x = y$ and this intersects $x^2+y^2 = 4$ at $\left(\sqrt{2}, \sqrt{2}\right)$. Note that $f\left(\sqrt{2}, \sqrt{2}\right) = 4\sqrt{2} + 6$. This is a maximum because $f(2, 0) = f(0, 2) = 8$ which is smaller.

Thus the minimum is $A = 8$ and so:

$$\int_C f(x, y)\, ds \geq \int_c 8\, ds = 8 \int_c 1\, ds = 8(\text{Length}) = 8\pi \approx 25.1327 > 25$$