MATH 241 Sections 03** Final Spring 2021

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups.
   Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. Given the two vectors:

$$\bar{\boldsymbol{u}} = 2\,\hat{\imath} + 3\,\hat{\jmath} - 5\,\hat{k} \text{ and } \bar{\boldsymbol{v}} = 5\,\hat{\imath} + 0\,\hat{\jmath} - 7\,\hat{k}$$

(a) Find $\bar{\boldsymbol{u}} \times \bar{\boldsymbol{v}}$. [5 pts]

Solution:

We have:

$$\bar{\boldsymbol{u}} \times \bar{\boldsymbol{v}} = -21\,\hat{\imath} - 11\,\hat{\jmath} - 15\,\hat{k}$$

(b) Find $\text{Proj}_{\bar{\boldsymbol{v}}}\bar{\boldsymbol{u}}$. [10 pts]

Solution:

We have:

$$\begin{aligned} \text{Proj}_{\bar{\boldsymbol{v}}}\bar{\boldsymbol{u}} &= \frac{\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}}}{\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}}}\bar{\boldsymbol{v}} \\ &= \frac{45}{74}(5\,\hat{\imath} + 0\,\hat{\jmath} - 7\,\hat{k}\,) \end{aligned}$$

(c) Find a vector of length 42 pointing in the same direction as $\bar{\boldsymbol{u}}$. [5 pts]

Solution:

Just make it unit length and then lengthen it:

$$42\frac{\bar{\boldsymbol{u}}}{||\bar{\boldsymbol{u}}||} = 42\frac{2\,\hat{\imath} + 3\,\hat{\jmath} - 5\,\hat{k}}{\sqrt{38}}$$

2. Consider the plane that passes through the point $P = (2, 5, 1)$ and includes the line with symmetric equation:

$$\frac{x-1}{2} = z + 3 \quad , \quad y = 1$$

(a) Find an equation for this plane in the form $ax + by + cz = d$ [20 pts]

Solution:

The point $Q = (1, 1, -3)$ is on the line and the vector $\bar{\boldsymbol{L}} = 2\,\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k}$ is the direction vector for the line. We can cross $\overline{PQ} \times \bar{\boldsymbol{L}}$ to get the normal vector:

$$\begin{aligned} \overline{PQ} &= -1\,\hat{\imath} - 4\,\hat{\jmath} - 4\,\hat{k} \\ \bar{\boldsymbol{L}} &= 2\,\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k} \\ \bar{\boldsymbol{N}} &= -4\,\hat{\imath} - 7\,\hat{\jmath} + 8\,\hat{k} \end{aligned}$$

An equation of the plane is then:

$$\begin{aligned} -4(x-1) - 7(y-1) + 8(z-(-3)) &= 0 \\ -4x + 4 - 7y + 7 + 8z + 24 &= 0 \\ -4x - 7y + 8z &= -35 \\ 4x + 7y - 8z &= 35 \end{aligned}$$

(b) Find y so that $(1, y, 3)$ is on this plane. [5 pts]

Solution:

We solve:

$$\begin{aligned} 4(1) + 7y - 8(3) &= 35 \\ 7y &= 55 \\ y &= 55/7 \end{aligned}$$

3. An object follows the path with parametrization:

$$\bar{\boldsymbol{r}}(t) = \cos t\,\hat{\imath} + 0\,\hat{\jmath} + \sin t\,\hat{k} \quad \text{for} \quad 0 \le t \le 2\pi$$

Floating in space is the paraboloid with equation:

$$z = \frac{2}{3}(x^2 + y^2)$$

(a) The object hits the paraboloid twice. Which point is first and which point is second? [10 pts]

Solution:

The object hits the paraboloid when:

$$\begin{aligned} \sin t &= \frac{2}{3}(\cos^2 t + 0^2) \\ 3\sin t &= 2 - 2\sin^2 t \\ 2\sin^2 t + 3\sin t - 2 &= 0 \\ (2\sin t - 1)(\sin t + 2) &= 0 \end{aligned}$$

This gives $\sin t = \frac{1}{2}$ which yields $t = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

This yields the two points:

First: $\bar{\boldsymbol{r}}(\pi/6) = \frac{1}{2}\,\hat{\imath} + 0\,\hat{\jmath} + \frac{\sqrt{3}}{2}\,\hat{k}$ so $\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$.

Second: $\bar{\boldsymbol{r}}(5\pi/6) = -\frac{1}{2}\,\hat{\imath} + 0\,\hat{\jmath} + \frac{\sqrt{3}}{2}\,\hat{k}$ so $\left(-\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$.

(b) Find the distance that the object travels within the paraboloid. [5 pts]

Solution:

The distance traveled is:

$$\begin{aligned} \text{Distance} &= \int_{\pi/6}^{5\pi/6} ||\bar{\boldsymbol{r}}'(t)||\,dt \\ &= \int_{\pi/6}^{5\pi/6} 1\,dt \\ &= \frac{2\pi}{3} \end{aligned}$$

4. Let R be the region bounded by the lines $x = 2$, $y = x - 1$ and $y = 1 - x$. [25 pts] Use the change of variables given by $u = x + y$ and $v = x - y$ to evaluate the following integral:

$$\iint_R \frac{1}{x+y}\, dA$$

You Should Evaluate Your Resulting Integral!

Solution:

We have $u + v = 2x$ so $x = \frac{1}{2}(u + v)$ and we have $u - v = 2y$ so $y = \frac{1}{2}(u - v)$.

The lines then become the following, bounding the new region S:

- $x = 2$ becomes $\frac{1}{2}(u - v) = 2$ or $v = 4 - u$.
- $y = x - 1$ becomes $x - y = 1$ or $v = 1$.
- $y = 1 - x$ becomes $x + y = 1$ or $u = 1$.

The Jacobian of the changer of variables is:

$$J(x, y) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

Thus we have:

$$\begin{aligned}
\iint_R \frac{1}{x+y}\, dA &= \iint_S \frac{1}{u} \left| -\frac{1}{2} \right| dA \\
&= \frac{1}{2} \int_1^3 \int_1^{4-u} \frac{1}{u}\, dv\, du \\
&= \frac{1}{2} \int_1^3 \left. \frac{v}{u} \right|_1^{4-u} du \\
&= \frac{1}{4} \int_1^3 \frac{4-u}{u} - \frac{1}{u}\, du \\
&= \frac{1}{4} \int_1^3 \frac{3}{u} - 1\, du \\
&= \frac{1}{4}(3\ln(u) - u)\Big|_1^3 \\
&= \frac{1}{4}(3\ln(3) - 3) - \frac{1}{4}(3\ln(1) - 1)
\end{aligned}$$

5. Define the function $f(x,y) = x^2y - 3xy^2 - y$.

(a) Find $\nabla f(1,2)$ and simplify. [5 pts]

Solution:

We have:

$$\nabla f(x,y) = (2xy - 3y^2)\,\hat{\imath} + (x^2 - 6xy - 1)\,\hat{\jmath}$$
$$\nabla f(1,2) = -8\,\hat{\imath} - 12\,\hat{\jmath}$$

(b) Explain why it is not possible to find a unit vector $\bar{\boldsymbol{u}}$ with $\bar{\boldsymbol{u}} \cdot \nabla f(1,2) = 15$. [10 pts]

Solution:

The expression $\bar{\boldsymbol{u}} \cdot \nabla f(1,2)$ equals the directional derivative of f in the direction of $\bar{\boldsymbol{u}}$ at the point $(1,2)$ and the magnitude of the gradient is the largest that this can be.

Since

$$||\nabla f(1,2)|| = \sqrt{208} < 15$$

we know that no direction will yield a value of 15.

(c) Explain why it is possible to find a unit vector $\bar{\boldsymbol{u}}$ with $\bar{\boldsymbol{u}} \cdot \nabla f(1,2) = 3$. [10 pts]

Solution:

If we want a unit vector $\bar{\boldsymbol{u}}$ with $\bar{\boldsymbol{u}} \cdot \nabla f(1,2) = 3$ this means if θ is the angle between $\bar{\boldsymbol{u}}$ and $\nabla f(1,2)$ then we need:

$$\begin{aligned}
\bar{\boldsymbol{u}} \cdot \nabla f(1,2) &= 3 \\
||\bar{\boldsymbol{u}}||\,||\nabla f(1,2)|| \cos\theta &= 2 \\
(1)(\sqrt{208}) \cos\theta &= 2 \\
\cos\theta &= \frac{2}{\sqrt{208}}
\end{aligned}$$

Since this value is between -1 and 1 we can certainly find such a θ.

6. Find and categorize (as relative max, relative min, or saddle points) all five critical points for the function: [25 pts]

$$f(x, y) = x^2y^2 - x^2 - 4y^2$$

Solution:

We find:

$$f_x(x, y) = 2xy^2 - 2x = 0$$
$$f_y(x, y) = 2x^2y - 8y = 0$$

The first yields:

$$2xy^2 - 2x = 0$$
$$2x(y^2 - 1) = 0$$
$$2x(y - 1)(y + 1) = 0$$

Thus either $x = 0$ or $y = 1$ or $y = -1$.

If $x = 0$ then the second yields $y = 0$ hence $(0, 0)$.

If $y = 1$ then the second yields $2x^2 - 8 = 0$ or $x = \pm 2$ hence $(2, 1)$ and $(-2, 1)$.

If $y = -1$ then the second yields $-2x^2 + 8 = 0$ or $x = \pm 2$ hence $(2, -1)$ and $(2, -1)$.

We then find the discriminant:

$$D(x, y) = (2y^2 - 2)(2x^2 - 8) - (4xy)^2$$

Then:

- $D(0, 0) = +$ and $f_{xx}(0, 0) = -$ hence a relative maximum.
- $D(2, 1) = -$ hence a saddle point.
- $D(-2, 1) = -$ hence a saddle point.
- $D(2, -1) = -$ hence a saddle point.
- $D(2, -1) = -$ hence a saddle point.

7. Evaluate the integral: [20 pts]

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

You Should Evaluate Your Resulting Integral!

Solution:

If we re-iterate in polar we get:

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \int_o^{\sqrt{2}} \sqrt{r^2} r\, dr\, d\theta &= \int_{\pi/4}^{\pi/2} \int_o^{\sqrt{2}} r^2\, dr\, d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{3} r^3 \Big|_0^{\sqrt{2}}\, d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{3}(2\sqrt{2})\, d\theta \\
&= \frac{1}{3}(2\sqrt{2})\theta \Big|_{\pi/4}^{\pi/2} \\
&= \frac{1}{3}(2\sqrt{2})\frac{\pi}{4}
\end{aligned}$$

8. Suppose Σ is the portion of the paraboloid $y = 4 - x^2$ in the first octant and below the plane $z = 3$. Let C be the edge of Σ with clockwise orientation when viewed from out in the first octant looking towards the origin. [25 pts]

Apply Stokes' Theorem to the integral:

$$\int_C (x+z)\,dx + x^2\,dy + xy\,dz$$

Proceed until you have an iterated double integral with the integrand simplified.

You Should Not Evaluate Your Resulting Integral!

Solution:

By Stokes' Theorem we have:

$$\int_C (x+z)\,dx + x^2\,dy + xy\,dz = \iint_\Sigma \left[x\,\hat{\imath} - (y-1)\,\hat{\jmath} + 2x\,\hat{k}\right] \cdot \bar{\boldsymbol{n}}\,dS$$

Where Σ is as given, oriented towards the z-axis.

We parametrize Σ:

$$\bar{\boldsymbol{r}}(x,z) = x\,\hat{\imath} + (4-x^2)\,\hat{\jmath} + z\,\hat{k}$$
$$0 \le x \le 2$$
$$0 \le z \le 3$$

Then:

$$\bar{\boldsymbol{r}}_x = 1\,\hat{\imath} - 2x\,\hat{\jmath} + 0\,\hat{k}$$
$$\bar{\boldsymbol{r}}_z = 0\,\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k}$$
$$\bar{\boldsymbol{r}}_x \times \bar{\boldsymbol{r}}_z = -2x\,\hat{\imath} - 1\,\hat{\jmath} + 0\,\hat{k}$$

Which matches Σ's orientation. Thus we have:

$\iint_\Sigma \left[x\,\hat{\imath} - (y-1)\,\hat{\jmath} + 2x\,\hat{k}\right] \cdot \bar{\boldsymbol{n}}\,dS$

$$= \iint_R \left[x\,\hat{\imath} - ((4-x^2)-1)\,\hat{\jmath} + 2x\,\hat{k}\right] \cdot \left(-2x\,\hat{\imath} - 1\,\hat{\jmath} + 0\,\hat{k}\right)\,dA$$
$$= \int_0^2 \int_0^3 -2x^2 + (4-x^2) - 1\,dy\,dx$$
$$= \int_0^2 \int_0^3 -3x^2 + 3\,dy\,dx$$

9. Define the vector field $\bar{\boldsymbol{F}}(x, y) = 4x\,\hat{\imath} + 6xy\,\hat{\jmath}$.

(a) If C is the counterclockwise triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, calculate $\int_C \bar{\boldsymbol{F}}(x, y) \cdot d\bar{\boldsymbol{r}}$. [10 pts]

You Should Evaluate Your Resulting Integral!

Solution:

By Green's Theorem we have:

$$\begin{aligned}
\int_C \bar{\boldsymbol{F}}(x, y) \cdot d\bar{\boldsymbol{r}} &= \iint_R 6y\,dA \\
&= \int_0^2 \int_0^{2x} 6y\,dy\,dx \\
&= \int_0^2 3y^2 \Big|_0^{2x}\,dx \\
&= \int_0^2 3(6x)^2\,dx \\
&= 36x^3 \Big|_0^2 \\
&= 288
\end{aligned}$$

(b) If C is the line segment from $(0, 0)$ to $(2, 4)$, calculate $\int_C \bar{\boldsymbol{F}} \cdot d\bar{\boldsymbol{r}}$. [10 pts]

You Should Evaluate Your Resulting Integral!

Solution:

We parametrize C by $\bar{\boldsymbol{r}}(t) = t\,\hat{\imath} + 2t\,\hat{\jmath}$ for $0 \le t \le 2$ and then $\bar{\boldsymbol{r}}'(t) = 1\,\hat{\imath} + 2\,\hat{\jmath}$ and:

$$\begin{aligned}
\int_C \bar{\boldsymbol{F}}(x, y) \cdot d\bar{\boldsymbol{r}} &= \int_0^2 (4(t)\,\hat{\imath} + 6(t)(2t)\,\hat{\jmath}) \cdot (1\,\hat{\imath} + 2\,\hat{\jmath})\,dt \\
&= \int_0^2 4t + 24t^2\,dt \\
&= 2t^2 + 8t^3 \Big|_0^2 \\
&= 72
\end{aligned}$$