Math 241 Spring 2022 Final Exam with Solutions

- Be sure your name, section number and problem number are on each answer sheet and that you have copied and signed the honor pledge on the first answer sheet.
- Follow the instructions as to which problem goes on which answer sheet. You may use the back of the answer sheets but if you do so, please write "See Back" or something similar on the bottom of the front so we know!
- No calculators or formula sheets are permitted.
- For problems with multiple parts, whether the parts are related or not, be sure to go on to subsequent parts even if there is some part you cannot do.
- Simplification of answers is not necessary. Please leave answers such as $5\sqrt{2}$ or 3π in terms of radicals and π and *do not convert to decimals.*

Please put problem 1 on answer sheet 1

1. (a) Suppose $\mathbf{u} = 3\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$ and $\mathbf{v} = 0\,\hat{\boldsymbol{\imath}} + 5\,\hat{\boldsymbol{\jmath}} + 7\,\hat{\boldsymbol{k}}$. Simplify the single expression: [15 pts]

$$(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v})$$

Solution:
We have:

$$\mathbf{u} \cdot \mathbf{v} = 3$$

And we have:

$$\mathbf{u} \times \mathbf{v} = 19\hat{\boldsymbol{\imath}} - 21\hat{\boldsymbol{\jmath}} + 15\hat{\boldsymbol{k}}$$

Then:

$$(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v}) = 3(19\hat{\boldsymbol{\imath}} - 21\hat{\boldsymbol{\jmath}} + 15\hat{\boldsymbol{k}})$$

(b) Write down a parameterization $\mathbf{r}(t)$ of the circle of radius 2 in the plane $z = 5$ and centered at $(1, 3, 5)$. [10 pts]

Solution:
One solution is:

$$\mathbf{r}(t) = (1 + 2\cos t)\hat{\boldsymbol{\imath}} + (3 + 2\sin t)\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$$
$$0 \le t \le 2\pi$$

Please put problem 2 on answer sheet 2

2. (a) Find the directional derivative of the function $f(x, y) = x^2y + \frac{x}{y}$ at the point $(1, 2)$ in the direction of the vector $5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$. [10 pts]

   Solution:

   We convert the vector to a unit vector:

$$\mathbf{u} = \frac{5}{\sqrt{34}}\hat{\boldsymbol{\imath}} - \frac{3}{\sqrt{34}}\hat{\boldsymbol{\jmath}}$$

   Then we have:

$$D_{\mathbf{u}}f(x, y) = \frac{5}{\sqrt{34}}\left(2xy + \frac{1}{y}\right) - \frac{3}{\sqrt{34}}\left(x^2 - \frac{x}{y^2}\right)$$

   And:

$$D_{\mathbf{u}}f(1, 2) = \frac{5}{\sqrt{34}}\left(2(1)(2) + \frac{1}{2}\right) - \frac{3}{\sqrt{34}}\left(1^2 - \frac{1}{2^2}\right)$$

   (b) Find the equation of the plane tangent to the graph of the function $f(x, y) = x^2y + x - y$ at the point $(1, 2, 1)$. [15 pts]

   Solution:

   If we put:

$$\begin{aligned} z &= x^2y + x - y \\ x^2y + x - y - z &= 0 \end{aligned}$$

   Then the normal vector may be found from the gradient of this function (call it g) of three variables:

$$\begin{aligned} \nabla g(x, y, z) &= (2xy + 1)\hat{\boldsymbol{\imath}} + (x^2 - 1)\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}} \\ \nabla g(1, 2, 1) &= 5\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}} \end{aligned}$$

   Thus the equation of the plane is:

$$3(x - 1) + 0(y - 2) - 1(z - 1) = 0$$

Please put problem 3 on answer sheet 3

3. (a) Find the distance between the two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$ given with symmetric equations: [10 pts]

$$\begin{aligned} \mathcal{L}_1 : \quad & x = \frac{y}{2} = \frac{z}{3} \\ \mathcal{L}_2 : \quad & x + 1 = \frac{y-1}{2} = \frac{z+5}{3} \end{aligned}$$

Solution:
Since the lines are parallel we can use any point on one line and the formula for the distance from a point to a line.
For example if we choose $Q = (0,0,0)$ on the first line and then $P = (-1,1,-5)$ on the second, and $\mathbf{L} = 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$ then we have:

$$\begin{aligned} \text{distance} &= \frac{||\overline{PQ} \times \mathbf{L}||}{||\mathbf{L}||} \\ &= \frac{||(1\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}) \times (1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}})||}{||1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}||} \\ &= \frac{|| - 13\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}||}{1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}||} \\ &= \frac{\sqrt{169+4+9}}{\sqrt{1+4+9}} \end{aligned}$$

(b) Suppose $\theta(t)$ is a function satisfying: [15 pts]

$$\theta(1) = \frac{\pi}{6} \quad \text{and} \quad \theta'(1) = 2$$

Find the tangent $\mathbf{T}$ vector at time $t = 1$ to the curve with parameterization:

$$\mathbf{r}(t) = \cos(\theta(t))\,\hat{\boldsymbol{\imath}} + \sin(\theta(t))\,\hat{\boldsymbol{\jmath}} + \theta(t)\,\hat{\boldsymbol{k}}$$

Solution:
We have:

$$\begin{aligned} \mathbf{r}'(t) &= \sin(\theta(t))\theta'(t)\hat{\boldsymbol{\imath}} - \cos(\theta(t))\theta'(t)\hat{\boldsymbol{\jmath}} + \theta'(t)\hat{\boldsymbol{k}} \\ \mathbf{r}'(1) &= \sin(\theta(1))\theta'(1)\hat{\boldsymbol{\imath}} - \cos(\theta(1))\theta'(1)\hat{\boldsymbol{\jmath}} + \theta'(1)\hat{\boldsymbol{k}} \\ &= \sin(\pi/6)(2)\hat{\boldsymbol{\imath}} - \cos(\pi/6)(2)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}} \\ &= 1\hat{\boldsymbol{\imath}} - \sqrt{3}\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}} \end{aligned}$$

Thus:

$$\mathbf{T}(1) = \frac{1\hat{\boldsymbol{\imath}} - \sqrt{3}\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}}{\sqrt{1+3+4}}$$

Please put problem 4 on answer sheet 4

4. Find and categorize all critical points for the function: [25 pts]

$$f(x, y) = x^2y + 4xy - 12y^2$$

Solution:

We have:

$$f_x(x, y) = 2xy + 4y = 0$$
$$f_y(x, y) = x^2 + 4x - 24y = 0$$

The first factors as:

$$2y(x + 2) = 0$$

Thus $y = 0$ or $x = -2$.

If $y = 0$ then the second yields $x^2 + 4x = 0$ or $x(x + 4) = 0$ so we get $(0, 0)$ and $(-4, 0)$.

If $x = -2$ then the second yields $4 - 8 - 24y = 0$ so we get $\left(-2, -\frac{1}{6}\right)$.

The discriminant is:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (2y)(-24) - (2x + 4)^2$$

So then:

$(0, 0)$ yields $D(0, 0) = -$ so it is a saddle point.

$(-4, 0)$ yields $D(-4, 0) = -$ so it is a saddle point.

$\left(-2, -\frac{1}{6}\right)$ yields $D\left(-2, -\frac{1}{6}\right) = \left(2\left(-\frac{1}{6}\right)\right)(-24) - (2(-2) + 4)^2 = 8$ and $f_{yy}\left(-2, -\frac{1}{6}\right) = -$ so it is a relative maximum.

Please put problem 5 on answer sheet 5

5. (a) Use the Fundamental Theorem of Line Integrals to evaluate: [10 pts]

$$\int_C (2x + y^2)\, dx + 2xy\, dy$$

Where C is the line segment from $(1, 3)$ to $(100, 200)$.

Solution:

The potential function is $f(x, y) = x^2 + xy^2$ and so we have:

$$\int_C (2x + y^2)\, dx + 2xy\, dy = f(100, 200) - f(1, 3) = (100^2 + (100)(200)^2) - (1^2 + (1)(3)^2)$$

(b) Consider the solid D that lies above the cone $z = \sqrt{3x^2 + 3y^2}$ and between the two spheres $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 3$. Assume the mass density at each point (x, y, z) is given by the distance of each point to the origin. Write down an iterated triple integral for the mass of D. [15 pts]

You Should Not Evaluate Your Resulting Integral!

Solution:

The mass is:

$$\iiint_D \sqrt{x^2 + y^2 + z^2}\, dv$$

Using spherical coordinates this is:

$$\int_0^{2\pi} \int_0^{\pi/6} \int_{\sqrt{2}}^{\sqrt{3}} \rho\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Please put problem 6 on answer sheet 6

6. Consider the region R in the xy-plane bounded by the circle $x^2 + y^2 = 2x$.

   (a) Treating R as a vertically simple region, set up an iterated double integral in rectangular coordinates that is equal to $\iint_R (x + y)\, dA$. [12 pts]
   You Should Not Evaluate Your Resulting Integral!
   Solution:
   We rewrite the circle as $y = \pm\sqrt{2x - x^2}$. This is defined on $0 \leq x \leq 2$ but this might take some work to figure out.
   We then have:

$$\iint_R (x + y)\, dA = \int_0^2 \int_{-\sqrt{2x-x^2}}^{+\sqrt{2x-x^2}} (x + y)\, dy\, dx$$

   (b) Set up an iterated double integral in polar coordinates that is equal to $\iint_R (x + y)\, dA$. [13 pts]
   You Should Not Evaluate Your Resulting Integral!
   Solution:
   If we rewrite the equation:

$$\begin{aligned} x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ x^2 - 2x + 1 + y^2 &= 1 \\ (x - 1)^2 + y^2 &= 1 \end{aligned}$$

   We see this is the circle $y = 2\cos\theta$ and so we have:

$$\int_R (x + y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

Please put problem 7 on answer sheet 7

7. Let Σ be the part of the vertical plane $x + 2y = 4$ in the first octant and below $z = 5$. Let C be the boundary/edge of Σ with clockwise orientation when viewed from the origin. Consider the integral: [25 pts]

$$\int_C z^2\,dx + x\,dy + z\,dz$$

Use Stokes' Theorem to rewrite the line integral as an iterated double integral.
You Should Not Evaluate Your Resulting Integral!

Solution:

We see that C is the edge of the surface Σ where Σ is given. By Stokes' Theorem we then have:

$$\begin{aligned}\int_C z^2\,dx + x\,dy + z\,dz &= \iint_\Sigma ((0-0)\,\hat{\boldsymbol{\imath}} - (0-2z)\,\hat{\boldsymbol{\jmath}} + (1-0)\,\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS \\ &= \iint_\Sigma (0\,\hat{\boldsymbol{\imath}} + 2z\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS\end{aligned}$$

Here Σ has orientation out into the first octant, induced by C.

We then parameterize Σ by:

$$\begin{gathered}\mathbf{r}(y,z) = (4-2y)\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}} \\ 0 \le y \le 2 \\ 0 \le z \le 6\end{gathered}$$

We have:

$$\begin{aligned}\mathbf{r}_y &= -2\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}} \\ \mathbf{r}_z &= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}} \\ \mathbf{r}_y \times \mathbf{r}_z &= 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\end{aligned}$$

These match the orientation of Σ and hence, letting R represent the set of inequalities:

$$\begin{aligned}\iint_\Sigma (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS &= +\iint_R (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot (1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}})\,dA \\ &= \iint_R 4z\,dA \\ &= \int_0^2 \int_0^5 4z\,dz\,dy\end{aligned}$$

Please put problem 8 on answer sheet 8

8. (a) Let D be the solid object above the xy-plane and below the paraboloid $z = 4 - x^2 - y^2$. [15 pts] Let Σ be the surface of D, oriented inwards. Evaluate the following integral:

$$\iint_\Sigma (2x\hat{\boldsymbol{\imath}} + 5x\hat{\boldsymbol{\jmath}} + 7z\hat{\boldsymbol{k}}) \cdot \mathbf{n}\, dS$$

Solution:

By the Divergence Theorem this equals:

$$-\iiint_D (2 + 0 + 7)\, dV$$

Where D is the solid given. If we parametrize in cylindrical:

$$\begin{aligned}
-\iiint_D (2 + 0 + 7)\, dV &= -9 \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta \\
&= -9 \int_0^{2\pi} \int_0^2 rz\Big|_0^{4-r^2} dr\, d\theta \\
&= -9 \int_0^{2\pi} \int_0^2 r(4 - r^2)\, dr\, d\theta \\
&= -9 \int_0^{2\pi} \int_0^2 4r - r^3\, dr\, d\theta \\
&= -9 \int_0^{2\pi} 2r^2 - \frac{1}{4}r^4\Big|_0^2 d\theta \\
&= -9 \int_0^{2\pi} 8 - 4\, d\theta \\
&= -9(4)(2\pi)
\end{aligned}$$

(b) Let C be the triangle with vertices $(0, 0)$, $(5, 1)$, and $(3, 1)$ with clockwise orientation. [10 pts] Use Green's Theorem to evaluate the following integral:

$$\int_C 6y\, dx + 15x\, dy$$

Solution:

By Green's Theorem this equals:

$$-\iint_R 15 - 6\, dA = -\iint_R 9\, dA = -9(\text{Area of } R) = -9\left(\frac{1}{2}(2)(1)\right)$$