Math 241 Exam 1 Fall 2014 Partial Solutions **Justin Wyss-Gallifent**

1. (a) If $\bar{a} = 3\,\hat{\imath} - 2\,\hat{\jmath} + 1\,\hat{k}$ and $\bar{b} = -4\,\hat{\imath} + 0\,\hat{\jmath} - 2\,\hat{k}$, find the sine and cosine of the angle θ between $\bar{a}$ and $\bar{b}$. [10 pts]

   Partial Solutions: Use the fact that $\bar{a}\cdot\bar{b} = ||\bar{a}||||\bar{b}||\cos\theta$ and $||\bar{a}\times\bar{b}|| = ||\bar{a}||||\bar{b}||\sin\theta$ to solve for $\cos\theta$ and $\sin\theta$.

   (b) Do the points $(0,0,0)$, $(0,3,4)$ and $(5,2,1)$ form a right triangle? Justify. [10 pts]

   Partial Solutions: Find the three distances and check if they satisfy the Pythagorean Theorem. Alternately one by one take each point, find the two vectors to the other two points and find the dot products each time to see if you get 0 for any of them. The first method is probably quicker.

2. (a) Find the equation of the plane containing the line $\frac{x}{2} = \frac{2-y}{3} = z$ and the point $(2,-1,3)$ and check whether this plane contains the origin. [10 pts]

   Partial Solutions: The direction vector for the line $\bar{L} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 1\,\hat{k}$ is parallel to the plane. Another vector parallel to the plane is the vector from $(0,2,0)$ (which is on the line) and $(2,-1,3)$, that being $\bar{a} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 3\,\hat{k}$. Cross these to get $\bar{N}$ and then form the plane equation. To check if the origin is on the plane plug in $(0,0,0)$ and check.

   (b) Provide a piecewise smooth parametrization of the triangle with vertices $(0,0)$, $(5,2)$ and $(4,7)$. [10 pts]

   Partial Solutions: This will have three different $\bar{r}(t)$. For each pair of points $P = (x_0, y_0, z_0)$ and Q, find $\bar{L} = \overrightarrow{PQ}$, call this $a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$ and then $\bar{r}(t) = (x_0 + at)\,\hat{\imath} + (y_0 + bt)\,\hat{\jmath} + (z_0 + ct)\,\hat{k}$ for $0 \le t \le 1$.

3. (a) Find the tangent and normal vectors for the curve $\bar{r}(t) = t^2\,\hat{\imath} + t\,\hat{\jmath}$ at $t = \sqrt{2}$. [10 pts]

   Partial Solutions: Just follow the formulas. The formula for $\bar{N}(t)$ isn't very pretty, make sure you plug $\sqrt{2}$ into $\bar{T}'(t)$ before finding the magnitude and dividing.

   (b) Find the value of a so that the curve $\bar{r}(t) = t\,\hat{\imath} + (2+3t)\,\hat{\jmath}$ for $0 \le t \le a$ has length 7. [10 pts]

   Partial Solutions: Take $\int_0^a ||\bar{v}(t)||\;dt$ and set it equal to 7, then solve for a.

4. (a) Sketch the sphere $x^2 + y^2 - 6y = -z^2 + 2z$. Mark at least two points with their coordinates. [5 pts]

   Partial Solutions: Move everything on the left and complete the square in y and z. The radius turns out to be $\sqrt{10}$. To get two points just add or subtract this radius to any of the coordinates of the center.

   (b) Find the equation of the sphere with center $(1,-2,1)$ and which also contains the origin. [5 pts]

   Partial Solutions: The radius is the distance from $(1,-2,1)$ to the origin. Then just stuff it in the formula.

   (c) Find the distance between the plane containing $(0,0,1)$, $(1,2,3)$ and $(0,-2,4)$ and the origin. [10 pts]

   Partial Solutions: If the points are called P, Q and R, find $\bar{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$ and then use the formula for distance from a point to a plane.

5. (a) Sketch $\bar{r}(t) = 2\sin(t)\,\hat{\imath} + 5\cos(t)\,\hat{\jmath} - 1\,\hat{k}$ for $0 \le t \le \frac{\pi}{2}$. Label the start and end points with their coordinates and indicate direction. [10 pts]

   Partial Solutions: This is a quarter-ellipse from $(0,5,-1)$ to $(2,0,-1)$. If it helps first draw it in the xy-plane from $(0,5)$ to $(2,0)$ and then shift down 1.

   (b) Let $\mathcal{L}$ be the line $\bar{r}(t) = (2-t)\,\hat{\imath} + t\,\hat{\jmath} + (3t+1)\,\hat{k}$ and let $\mathcal{P}$ be the plane $x + y + 2z = 10$.

   i. Find the point where $\mathcal{L}$ and $\mathcal{P}$ meet. [5 pts]

   Partial Solutions: Plug $x = 2-t$, $y = t$ and $z = 3t+1$ into the plane equation and solve for t. Then use $x = ...$ to find the point.

   ii. Show that the line is not perpendicular to the plane. [5 pts]

   Partial Solutions: This can be a bit confusing because as soon as people hear "perpendicular" they think "dot product" but that's not what's going on here. The normal vector for the plane is already perpendicular to the plane so if the line were perpendicular to the plane then the direction vector for the line and the normal vector would be parallel, meaning that's what needs to be checked.