Math 241 Exam 1 Fall 2018 Solutions **Justin Wyss-Gallifent**

1. (a) Find the center and radius of the sphere with equation $x^2 + 4x + y^2 + z^2 - 6x = 0$. [10 pts]
Solution: We complete the square:

$$x^2 + 4x + 4 + y^2 + z^2 - 6x + 9 = 13$$
$$(x+2)^2 + y^2 + (z-3)^2 = 13$$

The center is $(-2, 0, 3)$ and the radius is $\sqrt{13}$.

(b) Show that the points $(1, 2)$, $(2, 5)$ and $(-5, 4)$ form a right triangle. [10 pts]
Solution: The vector joining the first and second points is $2\,\mathbf{i} + 3\,\mathbf{j}$.
The vector joining the first and third points is $-6\,\mathbf{i} + 2\,\mathbf{j}$.
Since the $(2\,\mathbf{i} + 3\,\mathbf{j}) \cdot (-6\,\mathbf{i} + 2\,\mathbf{j}) = 0$, they form a right angle.

Note: This can also be done by finding the lengths of the sides and showing that the Pythagorean Theorem holds.

2. Find the point at which the line with symmetric equation [20 pts]

$$\frac{2-x}{3} = \frac{y}{4} \ , \ z = 5$$

meets the plane $x + 5y - 2z = 9$.

Solution: The line has $x = 2 - 3t$, $y = 4t$ and $z = 5$. If we plug these into the plane we see

$$\begin{aligned} (2-3t) + 5(4t) - 2(5) &= 9 \\ 17t &= 17 \\ t &= 1 \end{aligned}$$

Thus they meet at $x = 2 - 3(1) = -1$, $y = 4(1) = 4$ and $z = 5$.

Alternate Solution: We know $y = 4\left(\frac{2-x}{3}\right)$ so if we substitute into the plane equation:

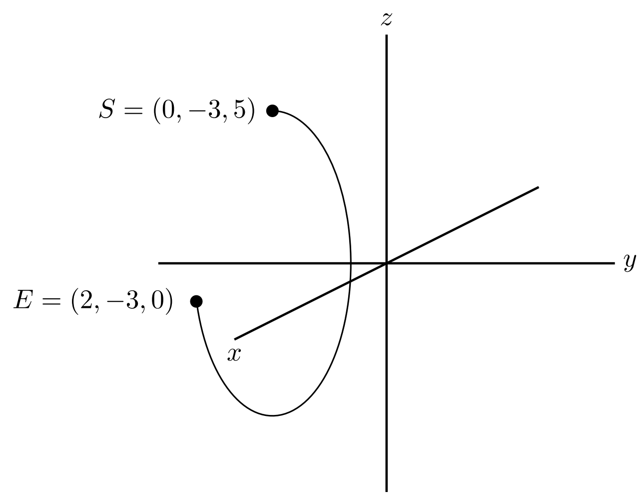
$$\begin{aligned} x + 5\left[4\left(\frac{2-x}{3}\right)\right] - 2(5) &= 9 \\ 3x + 40 - 20x - 30 &= 27 \\ -17x &= 17 \\ x &= -1 \end{aligned}$$

Then we have $y = 4\left(\frac{2-(-1)}{3}\right) = 4$ and $z = 5$.

3. (a) Sketch the curve with the following parametrization and label the start and end points with their coordinates: [10 pts]

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} - 3\,\mathbf{j} + 5\sin t\,\mathbf{k} \text{ with } \tfrac{\pi}{2} \le t \le 2\pi$$

Solution:



(b) Find a parametrization of the part of the parabola $y = 4 - x^2$ in the second quadrant along with the line segment joining the endpoints. [10 pts]

Solution: For the parabolic part we can use

$$\mathbf{r}(t) = t\,\mathbf{i} + (4 - t^2)\,\mathbf{j} \text{ with } -2 \le t \le 0$$

and for the linear part

$$\mathbf{r}(t) = (-2 + 2t)\,\mathbf{i} + (0 + 4t)\,\mathbf{j} \text{ with } 0 \le t \le 1$$

4. Suppose a curve has $\mathbf{v}(t) = 3t^2\,\mathbf{i} + 5t\,\mathbf{j} + (1-t)\,\mathbf{k}$.

(a) Find the tangent vector $\mathbf{T}(2)$. [10 pts]
Solution: We have:

$$\mathbf{T}(2) = \frac{\mathbf{v}(2)}{||\mathbf{v}(2)||} = \frac{12\,\mathbf{i} + 10\,\mathbf{j} - 1\,\mathbf{k}}{\sqrt{144+100=1}}$$

(b) Find the tangential component of acceleration at $t = 2$. [10 pts]
Solution: We have

$$\begin{aligned} \mathbf{v}(2) &= 12\,\mathbf{i} + 10\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{a}(t) &= 6t\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{a}(2) &= 12\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k} \end{aligned}$$

and so

$$\mathbf{a_T} = \frac{\mathbf{v}(2)\cdot\mathbf{a}(2)}{||\mathbf{v}(2)||} = \frac{144+50+1}{\sqrt{144+100+1}}$$

5. Find the distance between the point $(1, 2, 3)$ and the plane containing the point $(0, 0, 1)$ and the line $x = 1 + 2t$, $y = 5 - t$, $z = t$. [20 pts]

Solution: First we find $\mathbf{n}$ for the plane. The point $(1, 5, 0)$ is on the line and $(0, 0, 1)$ is on the plane and so $1\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k}$ is parallel to the plane. So is $\mathbf{L} = 2\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k}$. Then $\mathbf{n}$ can be the cross product:

$$\begin{gathered} 1\,\mathbf{i} + 5\,\mathbf{j} - 1\,\mathbf{k} \\ 2\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k} \\ \mathbf{n} = 4\,\mathbf{i} - 3\,\mathbf{j} - 11\,\mathbf{k} \end{gathered}$$

Then using $P = (0, 0, 1)$ and $Q = (1, 2, 3)$ we have $\overrightarrow{PQ} = 1\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$.

Then the distance is

$$\frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|(1)(4) + (2)(-3) + (2)(-11)|}{\sqrt{16 + 9 + 121}}$$