Math 241 Exam 1 Fall 2019 Solutions **Justin Wyss-Gallifent**

1. (a) Given the two vectors [10 pts]

$$\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} \text{ and } \mathbf{b} = 1\mathbf{i} + 0\mathbf{j} - 2\mathbf{k}$$

Find the cosine of the angle between them.

Solution: We have:

$$\begin{aligned} \cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \\ &= \frac{(2)(1)+(5)(0)+(0)(-2)}{\sqrt{4+25+0}\sqrt{1+0+4}} \end{aligned}$$

(b) Find the symmetric equation of the line passing through $(3, 2, 1)$ and perpendicular to the plane $4x + 2y - z = 1$. [10 pts]

Solution: If the line is perpendicular to the plane then $\mathbf{N} = 4\mathbf{i} + 2\mathbf{j} - 1\mathbf{k}$ may be used for $\mathbf{L}$. Thus:

$$\frac{x-3}{4} = \frac{y-2}{2} = \frac{z-1}{-1}$$

2. (a) Explain why the vector valued function $\mathbf{r}(t) = (t^2 - 4t)\,\mathbf{i} + 3\,\mathbf{j} + (t^3 - 12t)\,\mathbf{k}$ with $0 \leq t \leq 4$ is piecewise smooth but not smooth. [10 pts]

Solution: We find:

$$\mathbf{r}'(t) = (2t = 4)\,\mathbf{i} + 0\,\mathbf{j} + (3t^2 - 12)\,\mathbf{k}$$

First note that it's continuous.

We find that $r'(2) = 0$ and 2 is inside $[0, 4]$ which ruins smoothness.

However the VVF is smooth on $[0, 2]$ and on $[2, 4]$ and hence is piecewise smooth.

(b) Find the distance between the point $(1, 2, 3)$ and the plane $x - y + z = 0$. [10 pts]

Solution: We choose a point $P = (0, 0, 0)$ on the plane and let $Q = (1, 2, 3)$ then:

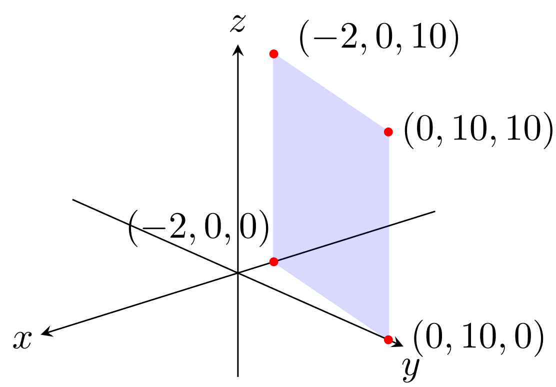
$$\overrightarrow{PQ} = 1\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$$

and so along with $\mathbf{N} = 1\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k}$ we have:

$$\begin{aligned} \text{dist} &= \frac{\overrightarrow{PQ} \cdot \mathbf{N}}{||\mathbf{N}||} \\ &= \frac{(1)(1) + (2)(-1) + (3)(1)}{\sqrt{1 + 1 + 1}} \end{aligned}$$

3. (a) Sketch the plane with equation $-10x + 2y = 20$. Label four points with their coordinates. [5 pts]

Solution: The x-intercept is at $(-2, 0, 0)$ and the y-intercept is at $(0, 10, 0)$. We extend in the z-direction:



(b) Consider the line with symmetric equation: [15 pts]

$$\frac{x-3}{2} = \frac{2-z}{5}, \quad y = 6$$

Find the equation of the plane containing the line as well as the point $(1, -1, 0)$. Write this in the form $ax + by + cz = d$.

Solution: Notice that the parametric form might help:

$$\begin{aligned} x &= 2t + 3 \\ y &= 0t + 6 \\ z &= -5t + 2 \end{aligned}$$

For the line we have $\mathbf{L} = 2\,\mathbf{i} + 0\,\mathbf{j} - 5\,\mathbf{k}$ which is parallel to the plane. If we let $P = (3, 6, 2)$ and $Q = (1, -1, 0)$ then $\overrightarrow{QP} = 2\,\mathbf{i} + 7\,\mathbf{j} + 2\,\mathbf{k}$ is also parallel.

Thus:

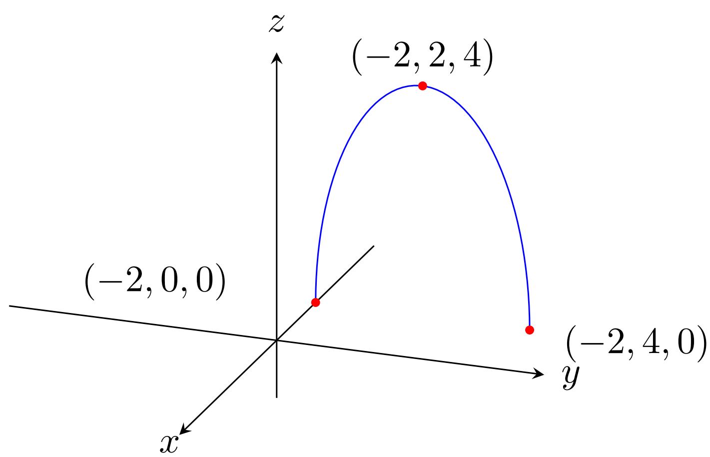
$$\mathbf{N} = \mathbf{L} \times \overrightarrow{QP} = 35\,\mathbf{i} - 14\,\mathbf{j} + 14\,\mathbf{k}$$

So the plane is:

$$\begin{aligned} 35(x-1) - 14(y+1) + 14(z-0) &= 0 \\ 35x - 14y + 14z &= 49 \end{aligned}$$

4. (a) Sketch the curve with parametrization $\mathbf{r}(t) = -2\,\mathbf{i} + (2+2\cos t)\,\mathbf{j} + 4\sin t\,\mathbf{k}$ for $0 \le t \le \pi$ and indicate the start point, middle point, and end point with their coordinates. [10 pts]

Solution:



(b) Find the parametrization(s) for the curve consisting of the part of $x^2 + y^2 = 9$ in the second quadrant along with the line segment joining the end points in a counterclockwise direction. [10 pts]

Solution: The part of the circle will have parametrization:

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \text{ with } \frac{\pi}{2} \le t \le \pi$$

while the line segment will have parametrization:

$$\mathbf{r}(t) = (-3+3t)\,\mathbf{i} + 3t\,\mathbf{j} \text{ with } 0 \le t \le 1$$

5. Given the vector-valued function $\mathbf{r}(t) = t^3\,\mathbf{i} - 5t^2\,\mathbf{j} + e^{2-t}\,\mathbf{k}$

(a) Set up but do not evaluate the integral for the length of the curve with $0 \leq t \leq 4$. [6 pts]

Solution: We have:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} - 10t\,\mathbf{j} - e^{2-t}\,\mathbf{k}$$

and so the length would be:

$$\text{Length} = \int_0^4 \sqrt{(3t^2)^2 + (-10t)^2 + (-e^{2-t})^2}\,dt$$

(b) Find the tangent vector at $t = 2$ [6 pts]

Solution: At $t = 2$ we have:

$$\mathbf{r}'(2) = 12\,\mathbf{i} - 20\,\mathbf{j} - 1\,\mathbf{k}$$

and so:

$$\mathbf{T}(2) = \frac{12\,\mathbf{i} - 20\,\mathbf{j} - 1\,\mathbf{k}}{\sqrt{144 + 400 + 1}}$$

(c) Find the tangential component of acceleration at $t = 2$. [8 pts]

Solution: We have:

$$\mathbf{a}(t) = 6t\,\mathbf{i} - 10\,\mathbf{j} + e^{2-t}\,\mathbf{k}$$
$$\mathbf{a}(2) = 12\,\mathbf{i} - 10\,\mathbf{j} + 1\,\mathbf{k}$$

Therefore:

$$a_T(2) = \frac{\mathbf{v}(2)\cdot\mathbf{a}(2)}{||\mathbf{v}(2)||} = \frac{(12)(12) + (-20)(-10) + (-1)(1)}{\sqrt{144 + 400 + 1}}$$