MATH 241 Sections 03** Exam 1 Key

1. Given the two vectors: [10 pts]

$$\bar{a} = 1\,\hat{\imath} + 8\,\hat{\jmath} + 4\,\hat{k}$$
$$\bar{b} = 4\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}$$

Rewrite $\bar{a}$ as $\bar{a} = \bar{a}_1 + \bar{a}_2$ where $\bar{a}_1$ is parallel to $\bar{b}$ and $\bar{a}_2$ is perpendicular to $\bar{b}$. In addition, verify that your final $\bar{a}_1$ and $\bar{a}_2$ are in fact perpendicular.

Solution:

We have:

$$\begin{aligned}
\bar{a}_1 &= \text{Proj}_{\bar{b}}\bar{a} \\
&= \frac{\bar{a}\cdot\bar{b}}{\bar{b}\cdot\bar{b}}\bar{b} \\
&= \frac{12}{24}(4\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}\,) \\
&= 2\,\hat{\imath} + 1\,\hat{\jmath} - 1\,\hat{k}
\end{aligned}$$

Then:

$$\bar{a}_2 = \bar{a} - \bar{a}_1 = -1\,\hat{\imath} + 7\,\hat{\jmath} + 5\,\hat{k}$$

We can see that they're perpendicular since:

$$\bar{a}_1 \cdot \bar{a}_2 = (2\,\hat{\imath} + 1\,\hat{\jmath} - 1\,\hat{k}\,) \cdot (-1\,\hat{\imath} + 7\,\hat{\jmath} + 5\,\hat{k}\,) = 0$$

2. Show that the set of points whose distance to $(-10, -5, 0)$ is three times their distance to $(6, 3, 0)$ form a sphere. Write the equation of this sphere in standard sphere form: [15 pts]

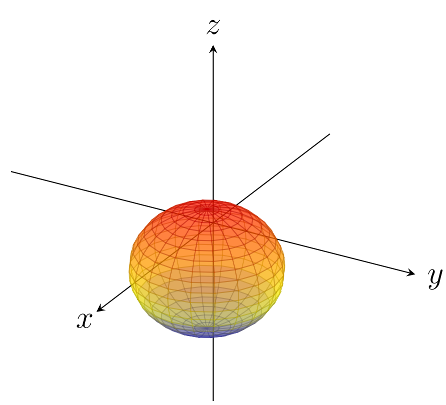
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

In addition, draw a reasonable sketch of the sphere.

Solution:

If (x, y, z) is such a point then we have:

$$\begin{aligned}
\sqrt{(x+10)^2 + (y+5)^2 + (z-0)^2} &= 3\sqrt{(x-6)^2 + (y-3)^2 + (z-0)^2} \\
x^2 + 20x + 100 + y^2 + 10y + 25 + z^2 &= 9(x^2 - 12x + 36 + y^2 - 6y + 9 + z^2) \\
8x^2 - 128x + 8y^2 - 64y + 8z^2 &= -280 \\
x^2 - 16x + y^2 - 8y + z^2 &= -35 \\
x^2 - 16x + 64 + y^2 - 8y + 16 + z^2 &= 45 \\
(x-8)^2 + (y-4)^2 + z^2 &= \left(\sqrt{45}\right)^2
\end{aligned}$$



3. Find the equation of the plane containing the point $(7, -2, 10)$ and containing the line with parametric equations [10 pts]

$$\begin{aligned} x &= 2 + 3t \\ y &= -2t \\ z &= 6 + t \end{aligned}$$

Write this plane in the form $ax + by + cz = d$.

Solution:

Assign $P = (7, -2, 10)$, pick $Q = (2, 0, 6)$ and $R = (5, -2, 7)$ on the line. Then assign:

$$\begin{aligned} \overline{PQ} &= -5\,\hat{\imath} + 2\,\hat{\jmath} - 4\,\hat{k} \\ \overline{PR} &= -2\,\hat{\imath} + 0\,\hat{\jmath} - 3\,\hat{k} \end{aligned}$$

The plane's normal vector can be calculated as:

$$\bar{n} = \overline{PQ} \times \overline{PR} = -6\,\hat{\imath} - 7\,\hat{\jmath} + 4\,\hat{k}$$

Note: We could have taken $\bar{L}$ from the line as well as $\overline{PQ}$ and done that cross product, too, or several other alternatives.

Then plane is then:

$$\begin{aligned} -6(x - 7) - 7(y + 2) + 4(z - 10) &= 0 \\ -6x - 7y + 4z &= 12 \\ 6x + 7y - 4z &= -12 \end{aligned}$$

4. Given the two planes: [15 pts]

$$\mathcal{P}_1 : 2x - y + 4z = 10$$
$$\mathcal{P}_2 : -2x + 5y + z = 6$$

Find the symmetric equations of the line where the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect.

Solution:

We first find a point on both planes. Adding the equations yields:

$$4y + 5z = 16$$

We choose $y = 4$ and $z = 0$. Then the first plane equation gives us $2x - 4 + 4(0) = 10$ or $x = 7$. Thus our point is $(7, 4, 0)$.

The direction vector for the line can be calculated by finding the cross product of the planes' normal vectors:

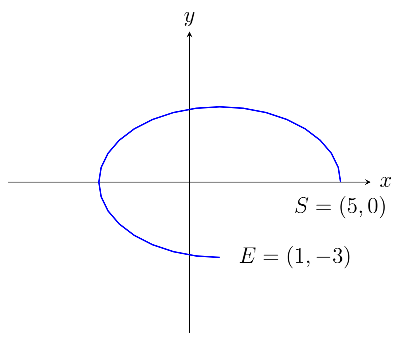
$$\bar{n}_1 = 2\,\hat{\imath} - 1\,\hat{\jmath} + 4\,\hat{k}$$
$$\bar{n}_2 = -2\,\hat{\imath} + 5\,\hat{\jmath} + 1\,\hat{k}$$
$$\bar{L} = \bar{n}_1 \times \bar{n}_2 = -21\,\hat{\imath} - 10\,\hat{\jmath} + 8\,\hat{k}$$

The symmetric equations are then:

$$\frac{x-7}{-21} = \frac{y-4}{-10} = \frac{z-0}{8}$$

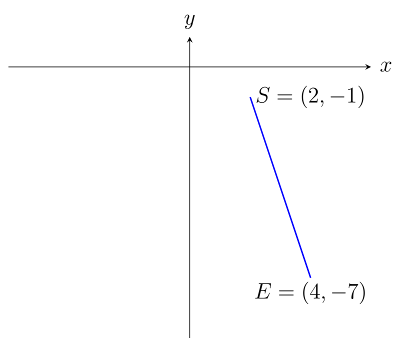
5. Sketch the three parametrizations given here. Indicate the start and end points with their coordinates and indicate the direction of the curve.

(a) $\bar{r}_1(t) = (1 + 4\cos t)\,\hat{\imath} + 3\sin t\,\hat{\jmath}$ [5 pts]
For: $0 \leq t \leq \frac{3\pi}{2}$
In 2D

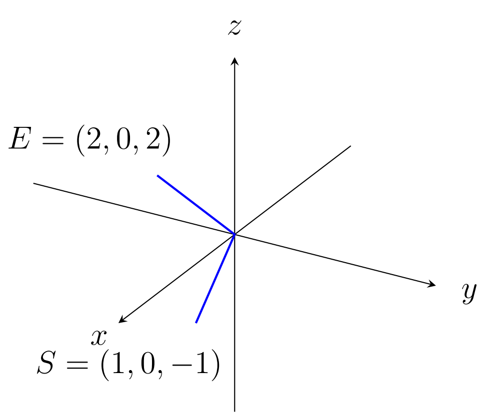


Solution:

(b) $\bar{r}_2(t) = (1 + t)\,\hat{\imath} + (2 - 3t)\,\hat{\jmath}$ [5 pts]
For: $1 \leq t \leq 3$
In 2D
Solution:



(c) $\bar{r}_3(t) = |t|\,\hat{\imath} + 0\,\hat{\jmath} + t\,\hat{k}$ [5 pts]
For: $-1 \leq t \leq 2$
In 3D
Solution:



6. Consider two curves with parametrizations: [10 pts]

$$\bar{r}_1(s) = (s^2+1)\,\hat{\imath} + (1-s)\,\hat{\jmath} + (2+3s)\,\hat{k}$$
$$\bar{r}_2(t) = (2+15t)\,\hat{\imath} - 3t\,\hat{\jmath} + (5+9t)\,\hat{k}$$

These curves meet at two points. Find these points.

Solution:

We equate the components:

$$s^2+1 = 2+15t$$
$$1-s = -3t$$
$$2+3s = 5+9t$$

The second gives us $s = 3t+1$ which we plug into the first:

$$(3t+1)^2+1 = 2+15t$$
$$9t^2+6t+2 = 2+15t$$
$$9t^2-9t = 0$$
$$9t(t-1) = 0$$

If $t=0$ then $s = 3(0)+1 = 1$ and if $t=1$ then $s = 3(1)+1 = 4$.

Both pairs satisfy the third equation.

When $t=0$ we get the point $\bar{r}_2(0) = 2\,\hat{\imath} + 0\,\hat{\jmath} + 5\,\hat{k}$ so $(2,0,5)$.

When $t=1$ we get the point $\bar{r}_2(1) = 17\,\hat{\imath} - 3\,\hat{\jmath} + 14\,\hat{k}$ so $(17,-3,14)$.

7. Given the parametrization $\bar{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} + t\,\hat{k}$,

(a) Find $\bar{v}(2)$ and $\bar{a}(2)$. [5 pts]

Solution: We have $\bar{v}(t) = 1\,\hat{\imath} + 2t\,\hat{\jmath} + 1\,\hat{k}$ and so $\bar{v}(2) = 1\,\hat{\imath} + 4\,\hat{\jmath} + 1\,\hat{k}$.
We have $\bar{a}(t) = 0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k}$ and so $\bar{a}(2) = 0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k}$.

(b) Find $\bar{T}(2)$. [5 pts]

Solution: We have:

$$\bar{T}(2) = \frac{\bar{v}(2)}{||\bar{v}(2)||} = \frac{1\,\hat{\imath} + 4\,\hat{\jmath} + 1\,\hat{k}}{\sqrt{18}} = \frac{1}{\sqrt{18}}\,\hat{\imath} + \frac{4}{\sqrt{18}}\,\hat{\jmath} + \frac{1}{\sqrt{18}}\,\hat{k}$$

(c) Find the tangential and normal components of acceleration, a_T and a_N, at $t = 2$. [5 pts]

Solution: We have:

$$a_T = \frac{\bar{v} \cdot \bar{a}}{||\bar{v}||} = \frac{8}{\sqrt{18}}$$

We have:

$$a_N = \frac{||\bar{v} \times \bar{a}||}{||\bar{v}||} = \frac{||-2\,\hat{\imath} + 0\,\hat{\jmath} + 2\,\hat{k}\,||}{\sqrt{18}} = \frac{\sqrt{8}}{\sqrt{18}} = \frac{2\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$$

(d) Using the fact that $\bar{a} = a_T\bar{T} + a_N\bar{N}$, find $\bar{N}(2)$. [5 pts]

Solution:

We have:

$$\begin{aligned}
\bar{a} &= a_T\bar{T} + a_N\bar{N} \\
0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k} &= \frac{8}{\sqrt{18}}\left(\frac{1\,\hat{\imath} + 4\,\hat{\jmath} + 1\,\hat{k}}{\sqrt{18}}\right) + \frac{2}{3}\bar{N} \\
0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k} &= \frac{8\,\hat{\imath} + 32\,\hat{\jmath} + 8\,\hat{k}}{18} + \frac{2}{3}\bar{N} \\
0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k} &= \frac{4}{9}\,\hat{\imath} + \frac{16}{9}\,\hat{\jmath} + \frac{4}{9}\,\hat{\jmath} + \frac{2}{3}\bar{N} \\
\frac{2}{3}\bar{N} &= -\frac{4}{9}\,\hat{\imath} - \frac{2}{9}\,\hat{\jmath} - \frac{4}{9}\,\hat{k} \\
\bar{N} &= -\frac{2}{3}\,\hat{\imath} + \frac{1}{3}\,\hat{\jmath} - \frac{2}{3}\,\hat{k}
\end{aligned}$$

(e) Write down the integral which represents the distance along the curve between $t = 0$ and $t = 2$. Without calculating this integral explain how you know the value is at least $2\sqrt{6}$. [5 pts]

Solution:

The integral is $\int_0^2 \sqrt{4t^2 + 2}\,dt$.
At $t = 0$ the curve is at $(0, 0, 0)$ and at $t = 2$ the curve is at $(2, 4, 2)$. These points are distance $\sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$ apart so the length is at least that.