Math 241 Exam 1 2021 **Justin Wyss-Gallifent**

1. These problems are independent of one another.

   (a) Explain in words why the vectors $\mathbf{a} = 5\,\mathbf{i} + 4\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = 2\,\mathbf{i} - 2\,\mathbf{j} + 1\,\mathbf{k}$ are not parallel to one another. [5 pts]

   Solution:

   They are not scalar multiples of one another.

   (b) Calculate $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = 5\,\mathbf{i} + 4\,\mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = 2\,\mathbf{i} - 2\,\mathbf{j} + 1\,\mathbf{k}$ and simplify. [5 pts]

   Solution:

   We have:

$$\begin{aligned}
\mathbf{a} &= 5\,\mathbf{i} + 4\,\mathbf{j} - 3\,\mathbf{k} \\
\mathbf{b} &= 2\,\mathbf{i} - 2\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{a} \times \mathbf{b} &= (4(1) - (-3)(-2))\,\mathbf{i} - ((5)(1) - (-3)(2))\,\mathbf{j} + ((5)(-2) - (4)(2))\,\mathbf{k} \\
&= -2\,\mathbf{i} - 11\,\mathbf{j} - 18\,\mathbf{k}
\end{aligned}$$

   (c) Find the tangential component of acceleration at $t = 2$ for the object following the path parametrized by: [15 pts]

$$\mathbf{r}(t) = (t^2 - t)\,\mathbf{i} + e^{2-t}\,\mathbf{j} + t\,\mathbf{k}$$

   Solution:

   We have:

$$\begin{aligned}
\mathbf{v}(t) &= (2t - 1)\,\mathbf{i} - e^{2-t}\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{a}(t) &= 2\,\mathbf{i} + e^{2-t}\,\mathbf{j} + 0\,\mathbf{k} \\
\mathbf{v}(2) &= 3\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k} \\
\mathbf{a}(2) &= 2\,\mathbf{i} + 1\,\mathbf{j} + 0\,\mathbf{k}
\end{aligned}$$

   Then:

$$\begin{aligned}
a_{\mathbf{t}} &= \frac{\mathbf{v}(2) \cdot \mathbf{a}(2)}{||\mathbf{v}(2)||} \\
&= \frac{5}{\sqrt{11}}
\end{aligned}$$

2. These problems are independent of one another.

   (a) Sketch the plane $x + 2z = 10$. [5 pts]

   Solution:

   (b) Find the vector equation of the line perpendicular to the plane containing the point $Q = (1, 2, 3)$ and containing the line $\mathbf{r}(t) = (2t + 1)\,\mathbf{i} + (3 - t)\,\mathbf{j} + (5t)\,\mathbf{k}$ [20 pts]

   Solution:

   The direction vector for the desired line can be the normal vector for the plane. A point on the given line is $P = (1, 3, 0)$ and the direction vector on the given line is $\mathbf{L} = 2\,\mathbf{i} - 1\,\mathbf{j} + 5\,\mathbf{k}$. To find the normal vector for the plane we take $\overline{PQ} \times \mathbf{L}$:

$$\begin{aligned} \overline{PQ} &= 0\,\mathbf{i} - 1\,\mathbf{j} + 3\,\mathbf{k} \\ \mathbf{L} &= 2\,\mathbf{i} - 1\,\mathbf{j} + 5\,\mathbf{k} \\ \overline{PQ} \times \mathbf{L} &= -2\,\mathbf{i} + 6\,\mathbf{j} + 2\,\mathbf{k} \end{aligned}$$

   Thus our desired line has equation:

$$\mathbf{r}(t) = (1 - 2t)\,\mathbf{i} + (2 + 6t)\,\mathbf{j} + (3 + 2t)\,\mathbf{k}$$

3. These problems are independent of one another.

(a) Explain in words why the following parametrization is not closed: [10 pts]

$$\mathbf{r}(t) = 2\sin(2t)\,\mathbf{i} - 3\cos(2t)\,\mathbf{j} \text{ for } 0 \leq t \leq 2\pi$$

Solution:

The parametrization follows an ellipse which traces over itself fully, therefore infinitely many times, therefore it is not closed.

(b) Write down a parametrization $\mathbf{r}(t) = ...$ of the graph of $y = \sin x$ for $0 \leq x \leq \pi$. [5 pts]

Solution:

We can use $\mathbf{r}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$.

(c) Draw the curve with parametrization: [10 pts]

$$\mathbf{r}(t) = (2+t)\,\mathbf{i} + (1-2t)\,\mathbf{j} + (1+t)\,\mathbf{k} \text{ for } 0 \leq t \leq 4$$

Solution:

Picture omitted here - the critical thing to note is that this is a straight line, so the easiest approach is to plot the starting and ending points and connect them directly.

4. Consider the object following the parametrization:

$$\mathbf{r}(t) = t^2\,\mathbf{i} + 3t\,\mathbf{j} + t\,\mathbf{k} \text{ for } t \geq 0$$

(a) At what time does the object hit the plane $y + 2z = 10$? [7 pts]

Solution:

We solve:

$$\begin{aligned} 3t + 2(t) &= 10 \\ 5t &= 10 \\ t &= 2 \end{aligned}$$

(b) At what time does the object hit the plane $x + 2y - z = 6$? [8 pts]

Solution:

We solve:

$$\begin{aligned} t^2 + 2(3t) - t &= 6 \\ t^2 + 5t - 6 &= 0 \\ (t+6)(t-1) &= 0 \end{aligned}$$

Since $t \geq 0$ we choose $t = 1$.

(c) Write down an expression (a calculation or an integral or whatever is appropriate) for the distance it travels between these two times. Do not simplify or evaluate whatever expression you write down. [10 pts]

Solution:

We have $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3\,\mathbf{j} + 1\,\mathbf{k}$ and so the distance would be:

$$\int_1^2 \sqrt{(2t)^2 + 9 + 1}\,dt$$