NAME (Neatly):

UID (Neatly):

Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

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(Except for doodling for stress relief.)

1. Given the vectors:

$$\bar{\boldsymbol{a}} = 2\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$$
$$\bar{\boldsymbol{b}} = 0\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$$
$$\bar{\boldsymbol{c}} = 3\hat{\boldsymbol{\imath}} + \alpha\hat{\boldsymbol{\jmath}} + \alpha\hat{\boldsymbol{k}}$$

(a) Calculate $\text{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}}$. [5 pts]

Solution:

We have:

$$\text{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}} = \frac{\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{b}}}{\bar{\boldsymbol{b}} \cdot \bar{\boldsymbol{b}}}\bar{\boldsymbol{b}} = -\frac{42}{45}(0\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}})$$

(b) Calculate all possible α so that $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{c}}$ are perpendicular. [5 pts]

Solution:

We would need:

$$\begin{aligned} \bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{c}} &= 0 \\ 6 + 6\alpha - 4\alpha &= 0 \\ 2\alpha &= -6 \\ \alpha &= -3 \end{aligned}$$

2. Find the distance between the point $(1, 2, 3)$ and the plane $2x + 3y - 2z = 10$. Simplify. [10 pts]

Solution:

The plane has $\bar{\boldsymbol{n}} = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$ and a point $Q = (5, 0, 0)$. If we let $P = (1, 2, 3)$ then the distance is:

$$
\begin{aligned}
\text{dist} &= \frac{\overline{PQ} \cdot \bar{\boldsymbol{n}}}{||\bar{\boldsymbol{n}}||} \\
&= \frac{(4\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}) \cdot (2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})}{||2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}||} \\
&= \frac{8}{\sqrt{17}}
\end{aligned}
$$

3. Find $a_T(1)$ for the curve with parameterization: [10 pts]

$$\bar{\boldsymbol{r}}(t) = t^3\hat{\boldsymbol{\imath}} + 5t\hat{\boldsymbol{\jmath}} + e^{2t-2}\hat{\boldsymbol{k}}$$

Solution:

We have:

$$
\begin{aligned}
\bar{\boldsymbol{v}}(t) &= 3t^2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 2e^{2t-2}\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{v}}(1) &= 3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{a}}(t) &= 6t\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 4e^{2t-2}\hat{\boldsymbol{k}} \\
\bar{\boldsymbol{a}}(1) &= 6\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}
\end{aligned}
$$

Then:

$$a_T(1) = \frac{\bar{\boldsymbol{a}}(1) \cdot \bar{\boldsymbol{v}}(1)}{||\bar{\boldsymbol{v}}(1)||} = \frac{26}{\sqrt{38}}$$

4. Suppose an object follows the path given by the parameterization:

$$\bar{\boldsymbol{r}}(t) = (3t-1)\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + t\hat{\boldsymbol{k}}$$

And given the plane with equation:

$$x + 2y - 3z = 10$$

(a) At what time does the object hit the plane? [5 pts]

Solution:

It hits when:

$$\begin{aligned} (3t-1) + 2(2t) - 3(t) &= 10 \\ 4t &= 11 \\ t &= 11/4 \end{aligned}$$

(b) At that instant, what is its velocity? [10 pts]

Solution:

We have:

$$\bar{\boldsymbol{v}}(t) = 3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$$

And so:

$$\bar{\boldsymbol{v}}(11/4) = 3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$$

5. Given the parameterization:

$$\bar{\boldsymbol{r}}(t) = \sin(\pi t)\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \quad \text{with} \quad -1 \leq t \leq 1$$

(a) Is the parameterization closed? Justify. [5 pts]

Solution:

We have:

$$\begin{aligned} \bar{\boldsymbol{r}}(-1) &= 0\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{r}}(1) &= 0\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \end{aligned}$$

Since these are equal, the parametrization is closed.

(b) Is the parameterization smooth? Justify. [10 pts]

Solution:

Observe that:

$$\bar{\boldsymbol{r}}'(t) = \pi\cos(\pi t)\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$

This exists and is continuous where $\bar{\boldsymbol{r}}(t)$ exists and it is never 0.

Thus it is smooth.

6. Find the symmetric equation of the line through the two points $(1, 4, -5)$ and $(8, 4, 10)$. [10 pts]

Solution:

We have direction vector:

$$\bar{\boldsymbol{L}} = 7\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 15\hat{\boldsymbol{k}}$$

Not necessary, but:

$$\begin{aligned} x &= 1 + 7t \\ y &= 4 \\ z &= -5 + 15t \end{aligned}$$

Symmetric:

$$\frac{x-1}{7} = \frac{z+5}{15} \quad , \quad y = 4$$

7. Find the equation of the plane containing the origin and the line with symmetric equation: [15 pts]

$$\frac{x-1}{2} = \frac{5-y}{6} = z$$

Solution:

The line has $\bar{\boldsymbol{L}} = 2\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$ which is parallel to the plane.

A point on the line is $(1, 5, 0)$ and so the vector from the origin to this point, $1\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ is parallel to the plane.

Thus the cross product works for $\bar{\boldsymbol{N}}$:

$$\bar{\boldsymbol{N}} = (2\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \times (1\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}) = -5\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 16\hat{\boldsymbol{k}}$$

Thus the plane has equation:

$$-5x - 1y + 16z = 0$$

8. Plot each of the following in 3D. On each, mark at least one point with its coordinates.

(a) $x^2 + y^2 + (z-2)^2 = 4$ [5 pts]

Solution:

Picture omitted.

This is a sphere of radius 2 centered at $(0,0,2)$. The most obvious point perhaps is the origin.

(b) $y = 3$ [5 pts]

Solution:

Picture omitted.

This is a plane stretching out in the x and z directions at $y = 3$.

(c) $\bar{\boldsymbol{r}}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}} \quad \text{for} \quad 0 \le t \le \pi$ [5 pts]

Solution:

Picture omitted.

This is a semicircle of radius 1 at $z = 2$ going from $(1,0,2)$ through $(0,1,2)$ to $(-1,0,2)$.