MATH241 Fall 2023 Exam 1 (Justin W-G) Solutions

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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

1. Write TRUE or FALSE in the box to the right. No justification is required. Unreadable or ambiguous answers will be marked as incorrect. [10 pts]

Solution:

Statement	TRUE/FALSE
$\bar{\boldsymbol{a}} \times \bar{\boldsymbol{b}}$ is a scalar.	FALSE
$\text{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}}$ is parallel to $\bar{\boldsymbol{b}}$.	TRUE
For any points P and Q we have $\lvert PQ\rvert = \lvert\lvert\overrightarrow{PQ}\rvert\rvert$.	TRUE
Planes with parallel normal vectors cannot meet.	FALSE
a_T is always non-negative.	FALSE

2. Given the point and vectors:

$$P = (4, -3, 1)$$
$$\bar{\boldsymbol{b}} = 5\,\hat{\boldsymbol{\imath}} - 3\,\hat{\boldsymbol{\jmath}} + 6\,\hat{\boldsymbol{k}}$$
$$\bar{\boldsymbol{c}} = 3\,\hat{\boldsymbol{\imath}} + \alpha\,\hat{\boldsymbol{\jmath}} + \alpha\,\hat{\boldsymbol{k}}$$

(a) Write down the equation of the sphere with center P and with radius $||\bar{\boldsymbol{b}}||$. [5 pts]

Solution:

This is just:

$$(x-4)^2 + (y+3)^2 + (z-1)^2 = 25 + 9 + 36$$

(b) For which α are $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{c}}$ perpendicular? [5 pts]

Solution:

We need:

$$\bar{\boldsymbol{b}} \cdot \bar{\boldsymbol{c}} = 0$$
$$15 - 3\alpha + 6\alpha = 0$$
$$\alpha = -5$$

3. Find the distance between the point $(4,2,5)$ and the line with symmetric equation: [10 pts]

$$\frac{x-1}{2} = y \quad \text{and} \quad z = 4$$

Simplify.

Solution:

A point on the line is $P = (1,0,4)$. If $Q = (4,2,5)$ then we have $\overrightarrow{PQ} = 3\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}$. Then since $\bar{\boldsymbol{L}} = 2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}$ we have

$$\overrightarrow{PQ} \times \bar{\boldsymbol{L}} = -1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$$

and then:

$$\frac{||\overrightarrow{PQ} \times \bar{\boldsymbol{L}}||}{||\bar{\boldsymbol{L}}||} = \frac{\sqrt{6}}{\sqrt{2^2+1^2+0^2}} = \frac{\sqrt{6}}{\sqrt{5}}$$

4. Find the equation of the plane containing the point $(3,4,-1)$ and perpendicular to the line with parameterization: [10 pts]

$$\bar{\boldsymbol{r}}(t) = (2t+1)\,\hat{\boldsymbol{\imath}} + (7-t)\,\hat{\boldsymbol{\jmath}} + 5t\,\hat{\boldsymbol{k}}$$

Write the result in the form $ax + by + cz = d$.

Solution:

The normal vector for the plane will be

$$\bar{\boldsymbol{n}} = \bar{\boldsymbol{L}} = 2\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}} + 5\,\hat{\boldsymbol{k}}$$

Thus the equation will be:

$$\begin{aligned} 2(x-3) - 1(y-4) + 5(z+1) &= 0 \\ 2x - y + 5z &= -3 \end{aligned}$$

5. Suppose an object follows the path given by the parameterization: [10 pts]

$$\bar{\boldsymbol{r}}(t) = t^3\,\hat{\boldsymbol{\imath}} + (t+1)\,\hat{\boldsymbol{\jmath}} - t\,\hat{\boldsymbol{k}}$$

What is the speed of the object when it hits the plane $x + 4y + 4z = 31$?

Solution:

The object hits the plane when:

$$\begin{aligned} t^3 + 4(t+1) + 4(-t) &= 31 \\ t^3 &= 27 \\ t &= 3 \end{aligned}$$

Then we have:

$$\begin{aligned} \bar{\boldsymbol{v}}(t) &= 3t^2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{v}}(3) &= 27\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\ ||\bar{\boldsymbol{v}}(t)|| &= \sqrt{27^2 + 1^2 + (-1)^2} \end{aligned}$$

6. Given the parameterization: [10 pts]

$$\bar{\boldsymbol{r}}(t) = 2\cos(t)\,\hat{\boldsymbol{\imath}} + 4\sin(t)\,\hat{\boldsymbol{\jmath}} + t\,\hat{\boldsymbol{k}}$$

Calculate the value of a_T, the tangential component of acceleration, at $t = \frac{\pi}{3}$.

Solution:

We have:

$$\begin{aligned} \bar{\boldsymbol{v}}(t) &= -2\sin(t)\,\hat{\boldsymbol{\imath}} + 4\cos(t)\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{v}}(\pi/3) &= -\sqrt{3}\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{a}}(t) &= -2\cos(t)\,\hat{\boldsymbol{\imath}} - 4\sin(t)\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{a}}(\pi/3) &= -1\,\hat{\boldsymbol{\imath}} - 2\sqrt{3}\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \end{aligned}$$

Then:

$$a_T = \frac{\bar{\boldsymbol{v}}(\pi/3) \cdot \bar{\boldsymbol{a}}(\pi/3)}{||\bar{\boldsymbol{v}}(\pi/3)||} = \frac{\sqrt{3} - 4\sqrt{3}}{\sqrt{3+4+1}}$$

7. Find the symmetric equation of the line segment joining the two points $(1,4,-5)$ and $(8,-4,10)$. [10 pts]

Solution:

We have:

$$\frac{x-1}{7} = \frac{y-4}{-8} = \frac{z+5}{15}$$

8. Suppose $\bar{\boldsymbol{r}}(t)$ for $0 \leq t \leq 3$ is a smooth parameterization which connects $(1,2)$ to $(5,5)$. You do not know exactly what the curve looks like! Explain in words how you know that: [10 pts]

$$\int_0^3 ||\bar{\boldsymbol{r}}'(t)||\, dt \geq 5$$

Solution:

The integral measures the length of the curve between the points $(1,2)$ and $(5,5)$. The distance between these points is $\sqrt{(1-5)^2+(2-5)^2} = 5$ so the length is at least that much.

9. Write down a non-closed parameterization of the circle $x^2+z^2=9$ in the plane $y=2$. [5 pts]

Solution:

One would be:

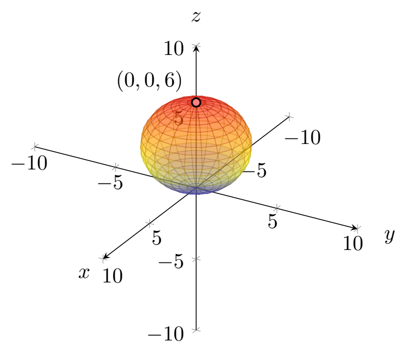
$$\bar{\boldsymbol{r}}(t) = 3\cos t\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 3\sin t\,\hat{\boldsymbol{k}} \quad \text{for} \quad 0 \leq t \leq 3\pi$$

This is not the only possible solution.

10. Plot each of the following. On each, mark at least one point with its coordinates.

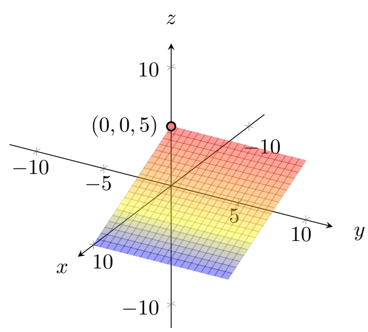
(a) In 3D: $x^2 + y^2 + (z-3)^2 = 9$ [5 pts]

Solution:



(b) In 3D: $x + 2z = 10$ [5 pts]

Solution:



(c) In 2D: $\bar{\boldsymbol{r}}(t) = (t^2+1)\,\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}}$ for $0 \le t \le 3$ [5 pts]

Solution: