1. (a) Suppose the points $(1,2,3)$ and $(3,8,7)$ are on opposite sides of a sphere. Write down the equation of the sphere. [10 pts]

Solution:

The center of the sphere is the midpoint $(2,5,5)$ and the radius is the distance from the midpoint to either of the given points (or half the distance between the given points) so the radius is $\sqrt{(1-2)^2+(2-5)^2+(3-5)^2}=\sqrt{14}$ so the equation is

$$(x-2)^2+(y-5)^2+(z-5)^2=14$$

(b) Suppose $\mathbf{a}=2\hat{\boldsymbol{\imath}}+8\hat{\boldsymbol{\jmath}}+1\hat{\boldsymbol{k}}$ and $\mathbf{b}=3\hat{\boldsymbol{\imath}}-4\hat{\boldsymbol{\jmath}}+7\hat{\boldsymbol{k}}$. Find the projection of $\mathbf{a}$ onto $\mathbf{b}$. [10 pts]

Solution:

We have:

$$\begin{aligned}\mathrm{Pr}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}\\ &= \frac{-19}{74}(3\hat{\boldsymbol{\imath}}-4\hat{\boldsymbol{\jmath}}+7\hat{\boldsymbol{k}})\end{aligned}$$

2. Find the equation of the plane containing the point $Q = (1, -1, 2)$ and containing the line with parametrization $\mathbf{r}(t) = (2t+1)\hat{\boldsymbol{\imath}} + (t-3)\hat{\boldsymbol{\jmath}} + (4-5t)\hat{\boldsymbol{k}}$. Write this in the form $ax + by + cz = d$. [20 pts]

Solution:

If we pick a point $P = (1, -3, 4)$ on the line and construct $\overrightarrow{PQ} = 0\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$ then with $\mathbf{L} = 2\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} - 5\hat{\boldsymbol{k}}$ then we can find the normal vector via:

$$\overrightarrow{PQ} \times \mathbf{L} = -8\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$$

then the plane is

$$\begin{aligned} -8(x-1) - 4(y+1) - 4(z-2) &= 0 \\ -8x - 4y - 4z &= -12 \end{aligned}$$

3. (a) Write down a parametrization of the straight line from $(1, -4, 3)$ to $(8, 4, 2)$. [5 pts]

Solution:

One answer would be $\mathbf{r}(t) = (1 + 7t)\hat{\boldsymbol{\imath}} + (-4 + 8t)\hat{\boldsymbol{\jmath}} + (3 - t)\hat{\boldsymbol{k}}$ for $0 \leq t \leq 1$.

(b) Find the distance between the parallel planes $2x + 3y + 10z = 10$ and $2x + 3y + 10z = 20$. [15 pts]

Solution:

If we pick an arbitrary point on the first plane, say $Q = (5, 0, 0)$, then we can find the distance from Q to the other plane. Since the other plane has $P = (10, 0, 0)$ and $\mathbf{N} = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 10\hat{\boldsymbol{k}}$ then we can find $\overrightarrow{PQ} = -5\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ and calculate:

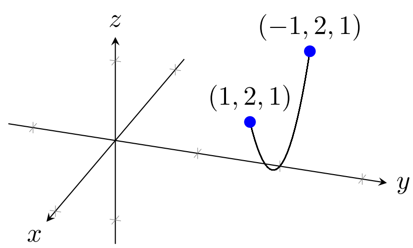
$$\text{distance} = \frac{|\overrightarrow{PQ} \cdot \mathbf{N}|}{||\mathbf{N}||} = \frac{|-10|}{\sqrt{113}}$$

4. Given the curve with parametrization

$$\mathbf{r}(t) = t\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}} \text{ for } -1 \le t \le 1.$$

(a) Sketch the curve. Mark the start and end points with their coordinates. [10 pts]

Solution:



(b) Is the parametrization closed or not? Justify. [5 pts]

Solution:

Since $\mathbf{r}(-1) = -1\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}+1\hat{\boldsymbol{k}}$ and $\mathbf{r}(1) = 1\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}+1\hat{\boldsymbol{k}}$ and these are different the parametrization is not closed.

(c) Is the parametrization smooth, piecewise smooth or neither? Justify. [5 pts]

Solution:

Since $\mathbf{r}'(t) = 1\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 2t\hat{\boldsymbol{k}}$ is continuous (everywhere) but never $\mathbf{0}$ the parametrization is smooth.

5. Consider the parametrization $\mathbf{r}(t) = t^2\hat{\boldsymbol{\imath}} + (1 - t^3)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$.

(a) Find $\mathbf{T}(1)$. [10 pts]

Solution:

We have $\mathbf{r}'(t) = 2t\hat{\boldsymbol{\imath}} - 3t^2\hat{\boldsymbol{k}} + 0\hat{\boldsymbol{k}}$ and $\mathbf{r}'(1) = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ and therefore

$$\mathbf{T}(1) = \frac{2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}}{\sqrt{13}}$$

(b) Find the tangential component of acceleration at $t = 1$. [10 pts]

Solution:

We already know $\mathbf{v}(1) = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ and we have $\mathbf{a}(t) = 2\hat{\boldsymbol{\imath}} - 6t\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ and so $\mathbf{a}(1) = 2\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ and hence

$$a_{\mathbf{T}} = \frac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{||\mathbf{v}(1)||} = \frac{22}{\sqrt{13}}$$