MATH 241 Sections Exam 1 Spring 2021 (JWG)

Exam Submission:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute.
2. Tag your problems! Please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use your notes and the textbook. You are permitted to use a calculator for basic arithmetic.
3. You are not permitted to use other resources. Thus no friends, internet, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. Consider the point and the line given here: [10 pts]

$$\begin{aligned} Point: &\quad (1,2,3) \\ Line: &\quad \bar{\boldsymbol{r}}(t) = t\,\hat{\boldsymbol{\imath}} + (2t+1)\,\hat{\boldsymbol{\jmath}} + 3t\,\hat{\boldsymbol{k}} \end{aligned}$$

Find the equation of the plane containing both the point and the line and write it in the form $ax + by + cz = d$.

Solution:

A point on the line is $P = (0,1,0)$. If $Q = (1,2,3)$ then $\overline{PQ} = 1\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}}$. For the normal vector we take the cross product:

$$\begin{aligned} \bar{\boldsymbol{L}} &= 1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}} \\ \overline{PQ} &= 1\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}} \\ \bar{\boldsymbol{N}} = \bar{\boldsymbol{L}} \times \overline{PQ} &= 3\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \end{aligned}$$

Thus the plane is:

$$\begin{aligned} 3(x-0) + 0(y-1) - 1(z-0) &= 0 \\ 3x - z &= 0 \end{aligned}$$

2. Show that the set of points in 2D which are equidistant from the points $(2, 3)$ and $(8, 10)$ form a line and find the parametric vector equation of the line. [10 pts]

Solution:

If (x, y) is such a point then:

$$\begin{aligned}
\sqrt{(x-2)^2+(y-3)^2} &= \sqrt{(x-8)^2+(y-10)^2} \\
x^2-4x+4+y^2-6y+9 &= x^2-16x+64+y^2-20y+100 \\
-4x-6y+13 &= -16x-20y+164 \\
12x+14y &= 151
\end{aligned}$$

For the parametric form we can for example set $x = t$ and then $y = \frac{151-12t}{14}$.

$$\bar{\boldsymbol{r}}(t) = t\,\hat{\boldsymbol{\imath}} + \left(\frac{151-12t}{14}\right)\hat{\boldsymbol{\jmath}}$$

3. Consider the plane and the line given here: [10 pts]

$$\begin{aligned} Plane: &\quad x - 2y + z = 18 \\ Line: &\quad \frac{x-1}{2} = y, z = 7 \end{aligned}$$

Show that the line and the plane are parallel and find the distance between them.

Solution:

The plane has $\bar{\boldsymbol{N}} = 1\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}$ and the line has $\bar{\boldsymbol{L}} = 2\,\hat{\boldsymbol{\imath}} + 1\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}$. Observe that $\bar{\boldsymbol{N}} \cdot \bar{\boldsymbol{L}} = 0$ so the normal vectors are perpendicular so the plane and line are parallel.

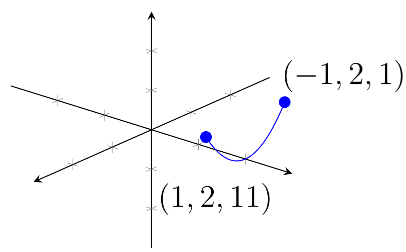
If we let $Q = (1, 0, 7)$ on the line and $P = (18, 0, 0)$ on the plane then:

$$\begin{aligned} \text{dist} &= \frac{\left|\overline{PQ} \cdot \bar{\boldsymbol{N}}\right|}{||\bar{\boldsymbol{N}}||} \\ &= \frac{\left|(-17\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 7\,\hat{\boldsymbol{k}}) \cdot (1\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}})\right|}{||1\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}||} \\ &= \frac{|-17 + 7|}{\sqrt{6}} \\ &= \frac{10}{\sqrt{6}} \end{aligned}$$

4. Given the curve with parameterization:

$$\bar{\boldsymbol{r}}(t) = t\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\boldsymbol{k}} \text{ for } -1 \le t \le 1.$$

(a) Sketch the curve. Mark the start and end points with their coordinates. [5 pts]
Solution:



(b) Is the parameterization closed or not? Justify. [5 pts]
Solution:
We have $\bar{\boldsymbol{r}}(-1) = -1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}$ and $\bar{\boldsymbol{r}}(1) = 1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}$. These are not the same so the parameterization is not closed.

(c) Is the parameterization smooth, piecewise smooth or neither? Justify. [5 pts]
Solution:
We have $\bar{\boldsymbol{r}}'(t) = 1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 2t\,\hat{\boldsymbol{k}}$ which is defined and continuous wherever $\bar{\boldsymbol{r}}$ is defined and is never $\bar{\mathbf{0}}$ so the parametrization is smooth.

5. Consider the curve and plane given here:

$$\begin{aligned} Curve: &\quad \bar{\boldsymbol{r}}(t) = t^2\,\hat{\boldsymbol{\imath}} + (2t+3)\,\hat{\boldsymbol{\jmath}} - t\,\hat{\boldsymbol{k}} \\ Plane: &\quad x - 2y + z = 18 \end{aligned}$$

(a) The curve intersects the plane twice. Find the two t-values and the two points. [5 pts]

Solution:

The intersection is where:

$$\begin{aligned} t^2 - 2(2t+3) - t &= 18 \\ t^2 - 5t - 24 &= 0 \\ (t-8)(t+3) &= 0 \end{aligned}$$

Thus they meet at $\bar{\boldsymbol{r}}(-3) = 9\,\hat{\boldsymbol{\imath}} - 3\,\hat{\boldsymbol{\jmath}} + 3\,\hat{\boldsymbol{k}}$ or $(9, -3, 3)$ and they meet at $\bar{\boldsymbol{r}}(8) = 64\,\hat{\boldsymbol{\imath}} + 19\,\hat{\boldsymbol{\jmath}} - 8\,\hat{\boldsymbol{k}}$ or $(64, 19, -8)$.

(b) Write down an integral for the distance traveled by the curve between the two intersection points. Do not evaluate. [5 pts]

Solution:

We have $\bar{\boldsymbol{r}}'(t) = 2t\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$ and so the distance is:

$$\int_{-3}^{8} \sqrt{4t^2 + 4 + 1}\,dt$$

(c) Determine whether the curve is traveling perpendicular to the plane at either intersection point. [5 pts]

Solution:

We have $\bar{\boldsymbol{r}}'(-3) = -6\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$ and $\bar{\boldsymbol{r}}'(8) = 16\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$, neither is parallel to $\bar{\boldsymbol{N}} = 1\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}} + 1\,\hat{\boldsymbol{k}}$ so nope.

6. Consider the curve with parameterization given here:

$$\bar{\boldsymbol{r}}(t) = (t^4 + t)\,\hat{\boldsymbol{\imath}} + 2t^3\,\hat{\boldsymbol{\jmath}} - (t + 8)\,\hat{\boldsymbol{k}}$$

Calculate each of the following:

(a) $\bar{\boldsymbol{v}}(1)$ [5 pts]

Solution:

We have $\bar{\boldsymbol{v}}(t) = (4t^3 + 1)\,\hat{\boldsymbol{\imath}} + 6t^2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$ so $\bar{\boldsymbol{v}}(1) = 5\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}$.

(b) $\bar{\boldsymbol{a}}(1)$ [5 pts]

Solution:

We have $\bar{\boldsymbol{a}}(t) = 12t^2\,\hat{\boldsymbol{\imath}} + 12t\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}$ so $\bar{\boldsymbol{a}}(1) = 12\,\hat{\boldsymbol{\imath}} + 12\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}$.

(c) $\bar{\boldsymbol{T}}(1)$ [5 pts]

Solution:

We have:

$$\begin{aligned}
\bar{\boldsymbol{T}}(1) &= \frac{\bar{\boldsymbol{v}}(1)}{||\bar{\boldsymbol{v}}(1)||} \\
&= \frac{5\,\hat{\boldsymbol{\imath}} + 6\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}}{\sqrt{25 + 36 + 1}} \\
&= \frac{5}{\sqrt{62}}\,\hat{\boldsymbol{\imath}} + \frac{6}{\sqrt{62}}\,\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{62}}\,\hat{\boldsymbol{k}}
\end{aligned}$$

(d) $a_{\bar{\boldsymbol{T}}}(1)$ [5 pts]

Solution:

We have:

$$\begin{aligned} a_{\bar{\boldsymbol{T}}}(1) &= \frac{\bar{\boldsymbol{v}}(1)\cdot\bar{\boldsymbol{a}}(1)}{|\bar{\boldsymbol{v}}(1)|} \\ &= \frac{(5)(12)+(6)(12)+(-1)(0)}{\sqrt{62}} \\ &= \frac{132}{\sqrt{62}} \end{aligned}$$

(e) $a_{\bar{\boldsymbol{N}}}(1)$ [5 pts]

Solution:

$$\begin{aligned} a_{\bar{\boldsymbol{N}}}(1) &= \frac{||\bar{\boldsymbol{v}}(1)\times\bar{\boldsymbol{a}}(1)||}{|\bar{\boldsymbol{v}}(1)|} \\ &= \frac{||12\,\hat{\boldsymbol{\imath}} - 12\,\hat{\boldsymbol{\jmath}} - 12\,\hat{\boldsymbol{k}}\,||}{\sqrt{62}} \\ &= \frac{12\sqrt{3}}{\sqrt{62}} \end{aligned}$$

(f) Use (b) through (e) to calculate $\bar{\boldsymbol{N}}(1)$. [5 pts]

Solution: We have:

$$\begin{aligned} \bar{\boldsymbol{a}}(1) &= a_{\bar{\boldsymbol{T}}}(1)\bar{\boldsymbol{T}}(1) + a_{\bar{\boldsymbol{N}}}(1)\bar{\boldsymbol{N}}(1) \\ 12\,\hat{\boldsymbol{\imath}} + 12\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} &= \frac{132}{\sqrt{62}}\left(\frac{5}{\sqrt{62}}\,\hat{\boldsymbol{\imath}} + \frac{6}{\sqrt{62}}\,\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{62}}\,\hat{\boldsymbol{k}}\right) + \frac{12\sqrt{3}}{\sqrt{62}}\bar{N}(1) \\ \frac{12\sqrt{3}}{\sqrt{62}}\bar{N}(1) &= \left(12\,\hat{\boldsymbol{\imath}} + 12\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}\right) - \frac{132}{\sqrt{62}}\left(\frac{5}{\sqrt{62}}\,\hat{\boldsymbol{\imath}} + \frac{6}{\sqrt{62}}\,\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{62}}\,\hat{\boldsymbol{k}}\right) \\ \bar{N}(1) &= \frac{\sqrt{62}}{12\sqrt{3}}(12\,\hat{\boldsymbol{\imath}} + 12\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}\,) - \frac{132}{12\sqrt{3}}\left(\frac{5}{\sqrt{62}}\,\hat{\boldsymbol{\imath}} + \frac{6}{\sqrt{62}}\,\hat{\boldsymbol{\jmath}} - \frac{1}{\sqrt{62}}\,\hat{\boldsymbol{k}}\right) \end{aligned}$$

7. Find the point on the sphere $(x-1)^2+(y+2)^2+(z-3)^2=16$ which is as far as possible from the point $(3,5,5)$. [10 pts]

Solution:

If the center of the sphere is $C=(1,-2,3)$ and $P=(3,5,5)$ then $\overline{CP} = 2\,\hat{\boldsymbol{\imath}} + 7\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\boldsymbol{k}}$ points from C to P.

The radius of the sphere is 4 so the vector:

$$4\frac{\overline{CP}}{||\overline{CP}||} = 4\left(\frac{2\,\hat{\boldsymbol{\imath}} + 7\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\boldsymbol{k}}}{\sqrt{57}}\right) = \frac{8}{\sqrt{57}}\,\hat{\boldsymbol{\imath}} + \frac{28}{\sqrt{57}}\,\hat{\boldsymbol{\jmath}} + \frac{8}{\sqrt{57}}\,\hat{\boldsymbol{k}}$$

Points from C to the point on the sphere closest to P. The negative of this points from C to the point on the sphere furthest from P so this point is then:

$$\left(1-\frac{8}{\sqrt{57}}\,,\,-2-\frac{28}{\sqrt{57}}\,,\,3-\frac{8}{\sqrt{57}}\right)$$