MATH241 Spring 2023 Exam 1 (Justin W-G) Solutions

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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

1. Write T for True or F for False in the box to the right. No justification is required. Unreadable or ambiguous letters will be marked as incorrect. [10 pts]

Solution:

Statement	T or F
$\bar{\boldsymbol{a}} \times (\bar{\boldsymbol{b}} \times \bar{\boldsymbol{c}})$ is a scalar.	F
The planes $2x + 3y - z = 0$ and $-4x - 6y + 2z = 100$ are parallel.	T
The spheres $x^2 + y^2 + z^2 = 100$ and $x^2 + y^2 + (z - 20)^2 = 100$ meet.	T
Acceleration always has positive magnitude.	F (*)
a_N is always non-negative.	T

(*) Because the magnitude can be 0, which is not positive.

2. Given the vectors:

$$\bar{\boldsymbol{a}} = 2\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$$
$$\bar{\boldsymbol{b}} = 0\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$$
$$\bar{\boldsymbol{c}} = 3\hat{\boldsymbol{\imath}} + \alpha\hat{\boldsymbol{\jmath}} + \alpha\hat{\boldsymbol{k}}$$

(a) Calculate $\mathrm{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}}$. [10 pts]

Solution:

We have:

$$\mathrm{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}} = \frac{\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{b}}}{\bar{\boldsymbol{b}} \cdot \bar{\boldsymbol{b}}}\bar{\boldsymbol{b}} = \frac{-42}{45}(0\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}) = -\frac{14}{15}(0\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}})$$

(b) Is it possible to choose α so that $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{c}}$ are parallel? Explain. [5 pts]

Solution:

No because for the $\hat{\boldsymbol{\imath}}$ components to work out we would need $\bar{\boldsymbol{c}} = \frac{3}{2}\bar{\boldsymbol{a}}$ but then we have $\alpha = \frac{3}{2}(6) = 9$ and also $\alpha = \frac{3}{2}(-4) = -6$ which doesn't work.

3. Find the distance between the point $(1, 2, 3)$ and the plane $5x + y - 10z = 10$. Simplify. [10 pts]

Solution:

A point on the plane is $P = (2, 0, 0)$. If $Q = (1, 2, 3)$ then we have $\overrightarrow{PQ} = -1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$ and since $\bar{\boldsymbol{n}} = 5\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}}$ the distance is:

$$\frac{|\overrightarrow{PQ} \cdot \bar{\boldsymbol{n}}|}{||\bar{\boldsymbol{n}}||} = \frac{|-33|}{\sqrt{126}} = \frac{33}{\sqrt{126}}$$

4. Find the equation of the plane containing the point $(3, 4, -1)$ and perpendicular to both of the planes: [10 pts]

$$2x + 4y - z = 0 \quad \text{and} \quad x + 5z = 6$$

Write the result in the form $ax + by + cz = d$.

Solution:

The first plane has $\bar{\boldsymbol{n}}_1 = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}}$ and the second has $\bar{\boldsymbol{n}}_2 = 1\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$. Our new plane will then have

$$\bar{\boldsymbol{n}} = \bar{\boldsymbol{n}}_1 \times \bar{\boldsymbol{n}}_2 = 20\hat{\boldsymbol{\imath}} - 11\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$$

So then it is:

$$20(x-3) - 11(y-4) - 4(z+1) = 0$$
$$20x - 11y - 4z = 20$$

5. Suppose an object follows the path given by the parameterization: [10 pts]

$$\bar{\boldsymbol{r}}(t) = t^3\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} - t\hat{\boldsymbol{k}}$$

The object intersects each of the planes $x + y + z = 8$ and $y - z = 10$ exactly once. Write down an integral for the distance the object travels between these two intersections. Do not evaluate.

Solution:

The first intersection occurs at:

$$\begin{aligned} t^3 + t - t &= 8 \\ t &= 2 \end{aligned}$$

The second intersection occurs at:

$$\begin{aligned} t + t &= 10 \\ t &= 5 \end{aligned}$$

So the distance traveled is:

$$\int_2^5 ||\bar{\boldsymbol{r}}'(t)||\, dt = \int_2^5 \sqrt{(3t^2)^3 + 1^2 + (-1)^2}\, dt$$

6. Assuming $\beta > 0$, for which values of β is the following parameterization smooth? Explain. [10 pts]

$$\bar{\boldsymbol{r}}(t) = \sin(\pi t)\hat{\boldsymbol{\imath}} + (t^2 - t)\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \quad \text{with} \quad 0 \le t \le \beta$$

Solution:

Observe that:

$$\bar{\boldsymbol{r}}'(t) = \pi\cos(\pi t)\hat{\boldsymbol{\imath}} + (2t - 1)\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$

This equals 0 when $t = \frac{1}{2}$ so the parameterization will be smooth for $\beta \le \frac{1}{2}$.

7. Find the symmetric equation of the line through the two points $(1, 4, -5)$ and $(8, 4, 10)$. [10 pts]

Solution:

We have $\bar{\boldsymbol{L}} = 7\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 15\hat{\boldsymbol{k}}$ so the result is:

$$\frac{x-1}{7} = \frac{z+5}{15}\,,\ y = 4$$

Note that this is not the only way to give the solution, for example if the other point was used.

8. Write down a parameterization of the line segment from $(1, 0, 5)$ to $(6, 3, 5)$. [5 pts]

Solution:

One would be:

$$\bar{\boldsymbol{r}}(t) = (5t+1)\hat{\boldsymbol{\imath}} + (3t+0)\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \quad \text{for} \quad 0 \le t \le 1$$

This is not the only possible solution.

9. Write down a parameterization of the circle $x^2 + z^2 = 9$ in the plane $y = 2$. [5 pts]

Solution:

One would be:

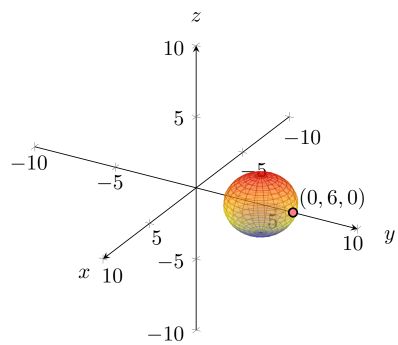
$$\bar{\boldsymbol{r}}(t) = 3\cos t\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\sin t\hat{\boldsymbol{k}} \quad \text{for} \quad 0 \le t \le 2\pi$$

This is not the only possible solution.

10. Plot each of the following in 3D. On each, mark at least one point with its coordinates.

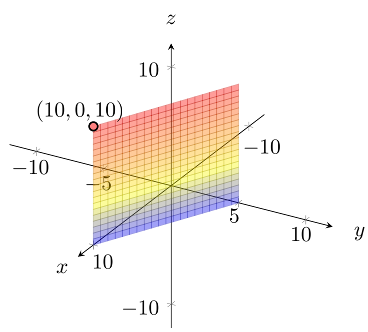
(a) $x^2 + (y-4)^2 + z^2 = 4$ [5 pts]

Solution:



(b) $x + 2y = 10$ [5 pts]

Solution:



(c) $\bar{\boldsymbol{r}}(t) = t\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}}$ for $0 \le t \le 3$ [5 pts]

Solution: