1. (a) Give the equation of a paraboloid opening down with vertex at (0, 0, 1). [5 pts]

**Solution:** There are various options but the most basic would be $z = 1 - x^2 - y^2$. You could also do things like $z = 1 - 4x^2 - 4y^2$ and so on, which just stretch the paraboloid vertically.

(b) Use tangent plane approximation to approximate $\sqrt{2.02^2 + 4.95^2}$. [15 pts]

**Solution:** The values 2.02 and 4.95 are close to 2 and 5. We assign $f(x, y) = \sqrt{x^2 + y}$ and $x_0 = 2$ and $y_0 = 5$ and then we’re looking for $f(2.02, 4.95)$. Then the formula states that for $(x, y)$ near $(x_0, y_0)$ we have:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(2.02, 4.95) \approx f(2, 5) + f_x(2, 5)(2.02 - 2) + f_y(2, 5)(4.95 - 5)$$

So we find:

$$f(2, 5) = \sqrt{2^2 + 5}$$
$$f(2, 5) = 3$$

and:

$$f_x(x, y) = \frac{1}{2}(x^2 + y)^{-1/2}(2x)$$
$$f_x(2, 5) = \frac{1}{2}(2^2 + 5)^{-1/2}(2)$$
$$f_x(2, 5) = \frac{2}{3}$$

and:

$$f_y(x, y) = \frac{1}{2}(x^2 + y)^{-1/2}(1)$$
$$f_y(2, 5) = \frac{1}{2}(2^2 + 5)^{-1/2}(1)$$
$$f_y(2, 5) = \frac{1}{6}$$

and so:

$$f(2.02, 4.95) \approx f(2, 5) + f_x(2, 5)(2.02 - 2) + f_y(2, 5)(4.95 - 5) + \frac{2}{3}(2.02 - 2) + \frac{1}{6}(4.95 - 5)$$
2. (a) Sketch the graph of the equation $z = -2 + \sqrt{x^2 + y^2}$. Label one point with its coordinates. [5 pts]

Name the shape.

**Solution:**

This is a cone:

(b) Suppose $f(x, y) = xy + y^2$. If $\bar{u}$ is a unit vector which makes an angle of $\pi/6$ with $\nabla f$ at $(2, -1)$, find $D_{\bar{u}}f(2, -1)$.

**Solution:** We can do:

$$D_{\bar{u}}f(x, y) = \bar{u} \cdot \nabla f(x, y)$$

$$D_{\bar{u}}f(x, y) = ||\bar{u}|| |\nabla f(x, y)|| \cos \theta$$

$$D_{\bar{u}}f(x, y) = ||\nabla f(x, y)|| \cos \theta$$

In this case we have

$$\nabla f(x, y) = f_x \hat{i} + f_y \hat{j}$$

$$= y \hat{i} + (x + 2y) \hat{j}$$

and so:

$$D_{\bar{u}}f(x, y) = ||\nabla f(x, y)|| \cos \theta$$

$$D_{\bar{u}}f(2, -1) = ||\nabla f(2, -1)|| \cos \theta$$

$$= || -1 \hat{i} + (2 + 2(-1)) \hat{j}|| \cos(\pi/6)$$

$$= || -1 \hat{i} + 0 \hat{j}|| \cos(\pi/6)$$

$$= \sqrt{3}/2$$
3. (a) Find a vector perpendicular to the graph of the function \( f(x, y) = x^2 y + 2x - y \) at \((-1, 2)\). [10 pts]

**Solution:** The graph of a function of two variables is a surface. In order to get a vector perpendicular to a surface we need to write the surface as the level surface for a function of three variables. So we change \( f(x, y) \) to \( z \) and rearrange:

\[
f(x, y) = x^2 y + 2x - y \\
z = x^2 y + 2x - y \\
x^2 y + 2x - y - z = 0
\]

Then we create a new function \( g(x, y, z) = x^2 y + 2x - y - z \) and take its gradient:

\[
g(x, y, z) = x^2 y + 2x - y - z \\
\nabla g(x, y, z) = (2xy + 2) \mathbf{i} + (x^2 - 1) \mathbf{j} - 1 \mathbf{k}
\]

\[
\nabla g(-1, 2, z) = -2 \mathbf{i} + 0 \mathbf{j} - 1 \mathbf{k}
\]

Note that we can find the \( z \) that goes with \((-1, 2)\) by doing \( f(-1, 2) \) but we don’t actually need to since there’s no \( z \) in \( \nabla g \).

(b) If \( z = x^2 + y \) where \( x = \frac{s}{t} \) and \( y = st \), use the Chain Rule to find \( \frac{\partial z}{\partial s} \) in terms of \( s \) and \( t \). [10 pts]

**Solution:**
The chart for the chain rule is:

\[
\begin{array}{c}
\text{z} \\
\downarrow \\
\text{x} \\
\uparrow \\
\text{s} \\
\downarrow \\
\text{y} \\
\uparrow \\
\text{t} \\
\end{array}
\]

And so we have:

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]

\[
= (2x) \left( \frac{1}{t} \right) + (1) \left( \frac{1}{t} \right)
\]

\[
= \left( 2 \left( \frac{s}{t} \right) \right) \left( \frac{1}{t} \right) + (1) \left( \frac{1}{t} \right)
\]
4. Find the maximum and minimum of \( f(x,y) = x^2 + y^2 \) where \((x, y)\) is restricted by the filled-in box shown here. Each tick mark is one unit.

\[ \text{Solution:} \]

First we find the critical points in the region. Since:

\[ f_x(x,y) = 2x \quad \text{and} \quad f_y(x,y) = 2y \]

there is only one critical point at \((0, 0)\). Since it's not in the region we throw it out.

Next we look at the edge. The edge is in four pieces so we must look at them all:

- **Top Edge**: Here \( y = 3 \) and \(-1 \leq x \leq 1\). So then \( f = x^2 + 9 \) which has a minimum of 9 when \( x = 0 \) and a maximum of 10 when \( x = \pm 1 \).
- **Bottom Edge**: Here \( y = 1 \) and \(-1 \leq x \leq 1\). So then \( f = x^2 + 1 \) which has a minimum of 1 when \( x = 0 \) and a maximum of 2 when \( x = \pm 1 \).
- **Left Edge**: Here \( x = -1 \) and \( 1 \leq y \leq 3 \). So then \( f = 1 + y^2 \) which has a minimum of 2 when \( y = 1 \) and a maximum of 10 when \( y = 3 \).
- **Right Edge**: Here \( x = 1 \) and \( 1 \leq y \leq 3 \). So then \( f = 1 + y^2 \) which has a minimum of 2 when \( y = 1 \) and a maximum of 10 when \( y = 3 \).

All together therefore we see a maximum of 10 and a minimum of 1.
5. Use Lagrange Multipliers to find the maximum value (there is no minimum value) of the function \( f(x, y) = xy^2 \) subject to the constraint \( x + y^2 = 2. \)

Note: Your system should have three solutions.

**Solution:**

We have the objective:
\[
f(x, y) = xy^2
\]

We have the constraint:
\[
g(x, y) = x + y^2
\]

We solve the system:
\[
\begin{align*}
y^2 &= \lambda(1) \quad (a) \quad \text{This is } f_x = \lambda g_x \\
2xy &= \lambda(2y) \quad (b) \quad \text{This is } f_y = \lambda g_y \\
x + y^2 &= 2 \quad (c) \quad \text{This is the constraint.}
\end{align*}
\]

From (a) we get \( \lambda = y^2 \) which we plug into (b) to get \( 2xy = y^2 (2y) \) which we factor and solve:
\[
\begin{align*}
2xy &= y^2 (2y) \\
2xy - 2y^3 &= 0 \\
2y(x - y^2) &= 0
\end{align*}
\]

So either \( y = - \) or \( x = y^2. \)

If \( y = 0 \) then (c) tells us \( x = 2 \) and we get the point \( (2, 0). \)

If \( x = y^2 \) then (c) tells us \( y^2 + y^2 = 2 \) so \( y = \pm 1. \) Since \( x = y^2 \) we get the points \( (1, 1) \) and \( (1, -1). \)

We test these points:

- \( f(2, 0) = 0 \)
- \( f(1, 1) = 1 \)
- \( f(1, -1) = 1 \)

So the maximum is 1 and the minimum is 0.