1. (a) Give the equation of a paraboloid opening down with vertex at $(0, 0, 1)$. [5 pts]

   Solution: There are various options but the most basic would be $z = 1 - x^2 - y^2$. You could also do things like $z = 1 - 4x^2 - 4y^2$ and so on, which just stretch the paraboloid vertically.

   (b) Use tangent plane approximation to approximate $\sqrt{2.02^2 + 4.95}$. [15 pts]

   Solution: The values 2.02 and 4.95 are close to 2 and 5. We assign $f(x, y) = \sqrt{x^2 + y}$ and $x_0 = 2$ and $y_0 = 5$ and then we're looking for $f(2.02, 4.95)$. Then the formula states that for (x, y) near (x_0, y_0) we have:

$$
\begin{aligned}
f(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
f(2.02, 4.95) &\approx f(2, 5) + f_x(2, 5)(2.02 - 2) + f_y(2, 5)(4.95 - 5)
\end{aligned}
$$

So we find:

$$
\begin{aligned}
f(2, 5) &= \sqrt{2^2 + 5} \\
f(2, 5) &= 3
\end{aligned}
$$

and:

$$
\begin{aligned}
f_x(x, y) &= \frac{1}{2}(x^2 + y)^{-1/2}(2x) \\
f_x(2, 5) &= \frac{1}{2}(2^2 + 5)^{-1/2}(2x) \\
f_x(2, 5) &= \frac{2}{3}
\end{aligned}
$$

and:

$$
\begin{aligned}
f_y(x, y) &= \frac{1}{2}(x^2 + y)^{-1/2}(1) \\
f_y(2, 5) &= \frac{1}{2}(2^2 + 5)^{-1/2}(1) \\
f_y(2, 5) &= \frac{1}{6}
\end{aligned}
$$

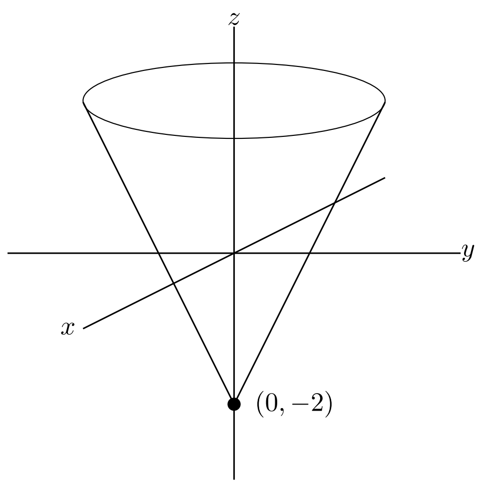
and so:

$$
f(2.02, 4.95) \approx f(2, 5) + f_x(2, 5)(2.02 - 2) + f_y(2, 5)(4.95 - 5)3 + \frac{2}{3}(2.02 - 2) + \frac{1}{6}(4.95 - 5)
$$

2. (a) Sketch the graph of the equation $z = -2 + \sqrt{x^2 + y^2}$. Label one point with its coordinates. Name the shape. [5 pts]

Solution:

This is a cone:



(b) Suppose $f(x, y) = xy + y^2$. If $\bar{u}$ is a unit vector which makes an angle of $\pi/6$ with ∇f at $(2, -1)$, find $D_{\bar{u}} f(2, -1)$. [15 pts]

Solution: We can do:

$$\begin{aligned} D_{\bar{u}} f(x, y) &= \bar{u} \cdot \nabla f(x, y) \\ D_{\bar{u}} f(x, y) &= ||\bar{u}|| ||\nabla f(x, y)|| \cos\theta \\ D_{\bar{u}} f(x, y) &= ||\nabla f(x, y)|| \cos\theta \end{aligned}$$

In this case we have

$$\begin{aligned} \nabla f(x, y) &= f_x\, \hat{\imath} + f_y\, \hat{\jmath} \\ &= y\, \hat{\imath} + (x + 2y)\, \hat{\jmath} \end{aligned}$$

and so:

$$\begin{aligned} D_{\bar{u}} f(x, y) &= ||\nabla f(x, y)|| \cos\theta \\ D_{\bar{u}} f(2, -1) &= ||\nabla f(2, -1)|| \cos\theta \\ &= || -1\, \hat{\imath} + (2 + 2(-1))\, \hat{\jmath}|| \cos(\pi/6) \\ &= || -1\, \hat{\imath} + 0\, \hat{\jmath}|| \cos(\pi/6) \\ &= \sqrt{3}/2 \end{aligned}$$

3. (a) Find a vector perpendicular to the graph of the function $f(x, y) = x^2y + 2x - y$ at $(-1, 2)$. [10 pts]

Solution: The graph of a function of two variables is a surface. In order to get a vector perpendicular to a surface we need to write the surface as the level surface for a function of three variables. So we change $f(x, y)$ to z and rearrange:

$$\begin{aligned} f(x, y) &= x^2y + 2x - y \\ z &= x^2y + 2x - y \\ x^2y + 2x - y - z &= 0 \end{aligned}$$

Then we create a new function $g(x, y, z) = x^2y + 2x - y - z$ and take its gradient:

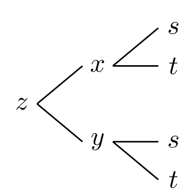
$$\begin{aligned} g(x, y, z) &= x^2y + 2x - y - z \\ \nabla g(x, y, z) &= (2xy + 2)\,\hat{\imath} + (x^2 - 1)\,\hat{\jmath} - 1\,\hat{k} \\ \nabla g(-1, 2, z) &= -2\,\hat{\imath} + 0\,\hat{\jmath} - 1\,\hat{k} \end{aligned}$$

Note that we can find the z that goes with $(-1, 2)$ by doing $f(-1, 2)$ but we don't actually need to since there's no z in ∇g.

(b) If $z = x^2 + y$ where $x = \frac{s}{t}$ and $y = st$, use the Chain Rule to find $\frac{\partial z}{\partial s}$ in terms of s and t. [10 pts]

Solution:

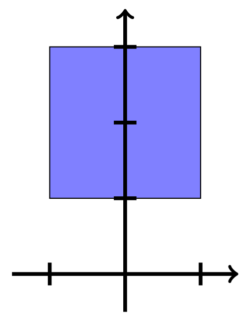
The chart for the chain rule is:



And so we have:

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)\left(\frac{1}{t}\right) + (1)\,(t) \\ &= \left(2\left(\frac{s}{t}\right)\right)\left(\frac{1}{t}\right) + (1)\,(t) \end{aligned}$$

4. Find the maximum and minimum of $f(x, y) = x^2 + y^2$ where (x, y) is restricted by the filled-in box shown here. Each tick mark is one unit. [20 pts]



Solution:

First we find the critical points in the region. Since:

$$f_x(x, y) = 2x \text{ and } f_y(x, y) = 2y$$

there is only one critical point at $(0, 0)$. Since it's not in the region we throw it out.

Next we look at the edge. The edge is in four pieces so we must look at them all:

- Top Edge: Here $y = 3$ and $-1 \leq x \leq 1$. So then $f = x^2 + 9$ which has a minimum of 9 when $x = 0$ and a maximum of 10 when $x = \pm 1$.
- Bottom Edge: Here $y = 1$ and $-1 \leq x \leq 1$. So then $f = x^2 + 1$ which has a minimum of 1 when $x = 0$ and a maximum of 2 when $x = \pm 1$.
- Left Edge: Here $x = -1$ and $1 \leq y \leq 3$. So then $f = 1 + y^2$ which has a minimum of 2 when $y = 1$ and a maximum of 10 when $y = 3$.
- Right Edge: Here $x = 1$ and $1 \leq y \leq 3$. So then $f = 1 + y^2$ which has a minimum of 2 when $y = 1$ and a maximum of 10 when $y = 3$.

All together therefore we see a maximum of 10 and a minimum of 1.

5. Use Lagrange Multipliers to find the maximum value (there is no minimum value) of the function $f(x,y) = xy^2$ subject to the constraint $x + y^2 = 2$. [20 pts]
Note: Your system should have three solutions.

Solution:

We have the objective:

$$f(x,y) = xy^2$$

We have the constraint:

$$g(x,y) = x + y^2$$

We solve the system:

$$\begin{aligned} y^2 &= \lambda(1) && (a) \quad \text{This is } f_x = \lambda g_x \\ 2xy &= \lambda(2y) && (b) \quad \text{This is } f_y = \lambda g_y \\ x + y^2 &= 2 && (c) \quad \text{This is the constraint.} \end{aligned}$$

From (a) we get $\lambda = y^2$ which we plug into (b) to get $2xy = y^2(2y)$ which we factor and solve:

$$\begin{aligned} 2xy &= y^2(2y) \\ 2xy - 2y^3 &= 0 \\ 2y(x - y^2) &= 0 \end{aligned}$$

So either $y = -$ or $x = y^2$.

If $y = 0$ then (c) tells us $x = 2$ and we get the point $(2,0)$.

If $x = y^2$ then (c) tells us $y^2 + y^2 = 2$ so $y = \pm 1$. Since $x = y^2$ we get the points $(1,1)$ and $(1,-1)$.

We test these points:

- $f(2,0) = 0$
- $f(1,1) = 1$
- $f(1,-1) = 1$

So the maximum is 1 and the minimum is 0.