Math 241 Exam 2 Fall 2018 Solutions **Justin Wyss-Gallifent**

1. (a) Sketch the graph of the surface $y = x^2$. Include some sense of size/position. Name the shape. [5 pts]
Solution: Picture omitted in solutions. Shape is a parabolic sheet with vertex on the z-axis, opening around the positive y-axis extending in the z-direction.

   (b) Write down the equation for the cylinder of radius 6 centered around the x-axis. [5 pts]
Solution: $y^2 + z^2 = 36$.

   (c) All together on one xy-plane sketch the level curves for $f(x,y) = x - y^2$ for $c = 0, 2, 4$. Label each with its value of c. [10 pts]
Solution: The level curves are:

$$\begin{gathered} c = 0 \text{ yields } x - y^2 = 0 \text{ or } x = y^2 \\ c = 2 \text{ yields } x - y^2 = 2 \text{ or } x = y^2 + 2 \\ c = 4 \text{ yields } x - y^2 = 4 \text{ or } x = y^2 + 4. \end{gathered}$$

Pictures omitted in solutions. These are parabolas opening to the right.

2. (a) Given $f(x,y) = x^2y + 3xy$, find the directional derivative of f in the direction of $2\,\mathbf{i} - 3\,\mathbf{j}$ at the point $(1,2)$. [12 pts]
Solution: The unit vector is $\mathbf{u} = \frac{2}{\sqrt{13}}\,\mathbf{i} - \frac{3}{\sqrt{13}}\,\mathbf{j}$ and so

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= \frac{2}{\sqrt{13}} f_x(x,y) - \frac{3}{\sqrt{13}} f_y(x,y) \\ D_{\mathbf{u}}f(x,y) &= \frac{2}{\sqrt{13}}(2xy + 3y) - \frac{3}{\sqrt{13}}(x^2 + 3x) \\ D_{\mathbf{u}}f(1,2) &= \frac{2}{\sqrt{13}}(10) - \frac{3}{\sqrt{13}}(4) \end{aligned}$$

   (b) Given $z = x^2y$ and $x = st^2$ and $y = t^3$ use the Chain Rule to calculate $\frac{\partial z}{\partial t}$. [8 pts]
Solution: We have:

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} + \frac{dy}{dt} \\ &= (2xy)(2st) + (x^2)(3t^2) \\ &= (2(st^2)(t^3))(2st) + ((st^2)^2)(3t^2) \end{aligned}$$

3. Find all points where the level surface $x - y^3 + z^2 = 0$ is parallel to the plane $3x - 9y + 24z = 10$. [20 pts]
Solution: Put $f(x,y,z) = x - y^3 + z^2$ and then find the gradient

$$\nabla f(x,y,z) = 1\,\mathbf{i} - 3y^2\,\mathbf{j} + 2z\,\mathbf{k}$$

For the surface to be parallel to the plane the normal vectors must be parallel so that

$$1\,\mathbf{i} - 3y^2\,\mathbf{j} + 2z\,\mathbf{k} \text{ must be a constant multiple of } \mathbf{n} = 3\,\mathbf{i} - 9\,\mathbf{j} + 24\,\mathbf{k}$$

Looking at the $\mathbf{i}$ components we see that we must have

$$3(1\,\mathbf{i} - 3y^2\,\mathbf{j} + 2z\,\mathbf{k}) = 3\,\mathbf{i} - 9\,\mathbf{j} + 24\,\mathbf{k}$$

Therefore we have: $3(-3y^2) = -9$ so $y = \pm 1$ and we must have $3(2z) = 24$ so $z = 4$.
When $y = 1$ and $z = 4$ we have $x = y^3 - z^2 = -15$ giving $(-15, 1, 4)$.
When $y = -1$ and $z = 4$ we have $x = y^3 - z^2 = -17$ giving $(-17, -1, 4)$.

4. Find all three of the critical points for the function $f(x,y) = x^2y - 2x^2 - y^2$. For each critical point calculate if it is a relative maximum, relative minimum or saddle point. [20 pts]
Solution: We find the partials and set equal to 0:

$$2xy - 4x = 0$$
$$x^2 - 2y = 0$$

The first factors as $2x(y-2) = 0$ so either $x = 0$ or $y = 2$. If $x = 0$ then the second yields $y = 0$ thus we have $(0,0)$. If $y = 2$ then the second yields $x = \pm 2$ thus we have $(\pm 2, 2)$.

We have $f_{xx} = 2y - 4$, $f_{yy} = -2$ and $f_{xy} = 2x$ and so $D(x,y) = (2y-4)(-2) - (2x)^2$. Then we check the points:

- $D(0,0) = +$ so then $f_{yy}(0,0) = -$ so it's a relative maximum.
- $D(2,2) = -$ so it's a saddle.
- $D(-2,2) = -$ so it's a saddle.

5. Use Lagrange Multipliers to find the maximum and minimum of $f(x,y) = 2x + xy$ with the constraint $x^2 + y^2 = 4$. Your system should have three solutions. [20 pts]
Solution: The constraint has $g(x,y) = x^2 + y^2 - 4$ and so the system to solve is

$$2 + y = \lambda(2x)$$
$$x = \lambda(2y)$$
$$x^2 + y^2 = 4$$

From the first, if $x = 0$ then $y = -2$ which satisfies the others so we have $(0,-2)$. If $x \neq 0$ then $\lambda = \frac{2+y}{2x}$ which we then plug into the second to get $x = \left(\frac{2+y}{2x}\right)$ or $x^2 = y^2 + y$. When we put this into the thiid we get $y^2 + 2y + y^2 = 4$ or $y^2 + y - 2 = 0$ which factors to $(y+2)(y-1) = 0$ giving $y = -2$ or $y = 1$. If $y = -2$ we have $x = 0$ yielding $(0,-2)$ again. If $y = 1$ we have $x^2 = 3$ yielding $x = \pm\sqrt{3}$ yielding $\left(\sqrt{3}, 1\right)$ and $\left(-\sqrt{3}, 1\right)$. We check these:

- $f(0,-2) = 0$
- $f\left(\sqrt{3}, 1\right) = 2\sqrt{3} + \sqrt{3}$ Maximum
- $f\left(-\sqrt{3}, 1\right) = -2\sqrt{3} - \sqrt{3}$ Minimum