MATH 241 Sections 03** Exam 2

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups.
   Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. Skech the following:

   (a) The surface $x^2 + (y-1)^2 = 1$. [5 pts]

   Partial Solution:

   Not included.

   (b) The surface $z = |x|$. [5 pts]

   Partial Solution:

   Not included.

   (c) The graph of $f(x, y) = 4 - \sqrt{x^2 + y^2}$. [5 pts]

   Partial Solution:

   Not included.

(d) The graph of $f(x, y) = 5 - y$. [5 pts]

Partial Solution:

Not included.

(e) The level surface of $f(x, y, z) = x - 2y + 3z - 10$ at the level $c = 2$. [5 pts]

Partial Solution:

Not included.

(f) The level curve of $f(x, y) = \frac{y}{|x-1|}$ at the level $c = 5$. [5 pts]

Partial Solution:

Not included.

2. Suppose an object moves along the curve with parametrization: [10 pts]

$$\bar{r}(t) = t^2\,\hat{\imath} - t^2\,\hat{\jmath} + t^3 k$$

At the instant when it passes through the point $(4, -4, 8)$ how fast is its distance from the origin changing? Assume x, y, z are measured in meters and time in seconds. Use the Chain Rule.

Partial Solution:

We have:

$$D(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

By the chain rule:

$$\begin{aligned}\frac{dD}{dt} &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(2t) + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(2t) + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(3t^2) \\ &= \frac{1}{2}(2t^4+t^6)^{-1/2}(2t^2)(2t) + \frac{1}{2}(2t^4+t^6)^{-1/2}(-2t^2)(-2t) + \frac{1}{2}(2t^4+t^6)^{-1/2}(2t^3)(3t^2)\end{aligned}$$

The instant we are interested in is $t = 2$:

$$\begin{aligned}\frac{dD}{dT}\Big|_{t=2} &= \frac{1}{2}(96)^{-1/2}(32) + \frac{1}{2}(96)^{-1/2}(32) + \frac{1}{2}(96)^{-1/2}(192) \\ &= \frac{128}{\sqrt{96}} \\ &= \frac{128}{4\sqrt{6}} \\ &= \frac{32}{\sqrt{6}}\end{aligned}$$

3. **Note:** Let A be the sum of the digits of your UID. Write down your UID and the value of A and mark them clearly. In the problem below, replace A by the appropriate value before proceeding. [10 pts]

Use tangent plane approximation to find an approximation of:

$$\sin\left(\frac{\pi}{6} + \frac{1}{A}\right)\sqrt{4 - \frac{1}{A}}$$

Simplify as much as possible but don't approximate your final answer.

Partial Solution:

We define:

$$f(x, y) = \sin(x)\sqrt{y}$$

We use the nice point as an anchor:

$$\left(\frac{\pi}{6}, 4\right)$$

Then we have:

$$\begin{array}{lcl} f(x,y) = \sin(x)\sqrt{y} & \Rightarrow & f(\pi/6, 4) = \sin(\pi/6)\sqrt{4} = \frac{1}{2}(2) = 1 \\ f_x(x,y) = \cos(x)\sqrt{y} & \Rightarrow & f_x(\pi/6, 4) = \cos(\pi/6)\sqrt{4} = \frac{\sqrt{3}}{2}(2) = \sqrt{3} \\ f_y(x,y) = \frac{1}{2}\sin(x)y^{-1/2} & \Rightarrow & f_y(\pi/6, 4) = \frac{1}{2}\sin(\pi/6)(4)^{-1/2} = \frac{1}{8} \end{array}$$

It then follows that:

$$\begin{aligned} f\left(\frac{\pi}{6} + \frac{\pi}{A}, \sqrt{4 - \frac{1}{A}}\right) &= 1 + \sqrt{3}\left(\frac{\pi}{6} + \frac{\pi}{A} - \frac{\pi}{6}\right) + \frac{1}{8}\left(4 - \frac{1}{A} - 4\right) \\ &= 1 + \frac{\pi\sqrt{3}}{A} - \frac{1}{8A} \end{aligned}$$

4. **Note:** Let B be the largest digit of your UID. Write down your UID and the value of B and mark them clearly. In the problem below, replace B by the appropriate value before proceeding.

Suppose the temperature in °C at (x, y) (measured in meters) is given by the function:

$$f(x,y) = x^2y + \frac{y^2}{B}$$

(a) If an object is at $\left(2, \frac{B}{2}\right)$ in what direction should it go in order to experience the maximum instantaneous temperature change? [5 pts]

Partial Solution:

We have:

$$\nabla f(x,y) = 2xy\,\hat{\imath} + \left(x^2 + \frac{2y}{B}\right)\hat{\jmath}$$

and so:

$$\nabla f\left(2, \frac{B}{2}\right) = 2B\,\hat{\imath} + 5\,\hat{\jmath}$$

(b) What will the maximum instantaneous temperature change be? Include units. [5 pts]

Partial Solution:

It will be:

$$||2B\,\hat{\imath} + 5\,\hat{\jmath}|| = \sqrt{4B^2 + 25}\ \text{°C/m}$$

5. **Note:** Let C be the second smallest nonzero digit of your UID. Write down your UID and the value of C and mark them clearly. In the problem below, replace C by the appropriate value before proceeding. [10 pts]

Consider the function:

$$f(x, y) = x(x^2 + y^2)^3$$

Find the vector equation of the line perpendicular to the graph of $f(x, y)$ at the point (C, C).

Partial Solution:

We set:

$$g(x, y, z) = x(x^2 + y^2)^3 - z$$

We have:

$$f(C, C) = C(C^2 + C^2)^3 = C(2C^2)^3 = 8C^7$$

Then we examine:

$$\begin{aligned}
\nabla g(x, y, z) &= \big((x^2 + y^2)^3 + 3x(x^2 + y^2)^2(2x)\big)\,\hat{\imath} + 3x(x^2 + y^2)^2(2y)\,\hat{\jmath} - 1\,\hat{k} \\
\nabla g(C, C, 8C^7) &= \big((C^2 + C^2)^3 + 3C(C^2 + C^2)^2(2C)\big)\,\hat{\imath} + 3C(C^2 + C^2)^2(2C)\,\hat{\jmath} - 1\,\hat{k} \\
&= \big(8C^6 + 6C^2(4C^4)\big)\,\hat{\imath} + 6C^2(4C^4)\,\hat{\jmath} - 1\,\hat{k} \\
&= 32C^6\,\hat{\imath} + 24C^6\,\hat{\jmath} - 1\,\hat{k}
\end{aligned}$$

The line is then:

$$\bar{r}(t) = (C + 32C^6 t)\,\hat{\imath} + (C + 24C^6 t)\,\hat{\jmath} + (8C^7 - t)\,\hat{k}$$

6. **Note:** Let D be the largest digit of your UID. Write down your UID and the value of D and mark them clearly. In the problem below, replace D by the appropriate value before proceeding. [10 pts]

Consider the function:

$$f(x,y) = \frac{1}{12}x^3 + 5Dxy + Dy^2$$

This function has two critical points. Find and classify.

Partial Solution:

To find the critical points we set:

$$\begin{aligned} f_x &= \frac{1}{4}x^2 + 5Dy = 0 \\ f_y &= 5Dx + 2Dy = 0 \end{aligned}$$

The second factors to $D(5x+2y) = 0$ and so $y = -\frac{5}{2}x$. Plugging into the first yields:

$$\begin{aligned} \frac{1}{4}x^2 + 5D\left(-\frac{5}{2}x\right) &= 0 \\ \frac{1}{4}x^2 - \frac{25D}{2}x &= 0 \\ x^2 - 50Dx &= 0 \\ x(x - 50D) &= 0 \end{aligned}$$

Thus either $x = 0$ or $x = 50D$.

The critical points are then:

$$(0,0) \quad \text{and} \quad (50D, -125D)$$

We have:

$$D(x,y) = \left(\frac{1}{2}x\right)(2D) - (5D)^2$$

and so:

- $D(0,0) = -25D^2 < 0$ which is a saddle point.
- $D(50D, -125D) = 25M^2 > 0$ and then $f_{yy}(50D, -125D) = 2D > 0$ which is a relative minimum.

7. **Note:** Let E be the sum of the leftmost three digits of your UID. Write down your UID and the value of E and mark them clearly. In the problem below, replace E by the appropriate value before proceeding. [10 pts]

Find the maximum of the function:

$$f(x, y) = Exy$$

Where (x, y) is constrained within the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, E)$.

Partial Solution:

The critical point is at $(0, 0)$ and $f(0, 0) = 0$.

Looking at the edges:

- On the left edge we have $f(x, y) = 0$ and so the function is always 0.
- On the bottom edge we have $f(x, y) = 0$ and so the function is always 0.
- On the diagonal edge we have $y = E - Ex$ and so the function is:

$$f = Ex(E - Ex)$$
$$f = -E^2x^2 + E^2x$$

We also have the restriction $0 \leq x \leq 1$.
This f is a parabola opening down which has its maximum at $x = \frac{1}{2}$ and has:

$$f = -E^2 \left(\frac{1}{2}\right)^2 + E^2 \left(\frac{1}{2}\right) = \frac{E^2}{4}$$

Thus the overall maximum is $\frac{E^2}{4}$.

8. **Note:** Let F be the sum of the leftmost four digits of your UID. Write down your UID and the value of F and mark them clearly. In the problem below, replace F by the appropriate value before proceeding. [10 pts]

Use Lagrange Multipliers to find the minimum value of the function:

$$f(x, y) = (x - F)^2 + y^2$$

Subject to the constraint:

$$x + y = 0$$

Partial Solution: The constraint has $g(x, y) = x + y$.

The system of equations is then:

$$\begin{aligned} 2(x - F) &= \lambda(1) \\ 2y &= \lambda(1) \\ x + y &= 0 \end{aligned}$$

The first two are equal to λ so set them equal to get:

$$\begin{aligned} 2(x - F) &= 2y \\ x - F &= y \end{aligned}$$

If we plug this into the constraint we get:

$$\begin{aligned} x + (x - F) &= 0 \\ 2x &= F \\ x &= \frac{F}{2} \end{aligned}$$

Then we have $y = x - F = -\frac{F}{2}$

Thus the single solution to the system is $\left(\frac{F}{2}, -\frac{F}{2}\right)$ and

$$f\left(\frac{F}{2}, -\frac{F}{2}\right) = \left(\frac{F}{2} - F\right)^2 + \left(-\frac{F}{2}\right)^2 = \frac{F^2}{2}$$