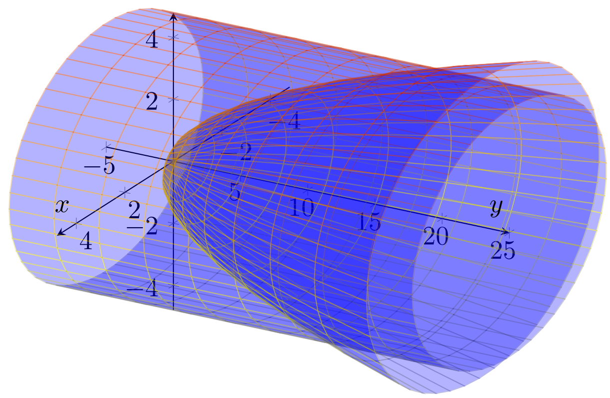
MATH241 Exam 2 Fall 2021 Justin Wyss-Gallifent

Solutions

1. Together in space, sketch the graphs of the two surfaces $x^2 + z^2 = 16$ and $y = x^2 + z^2$. Describe the intersection of these two surfaces. Use both words and equations to be precise. [10 pt]

   Solution:



They meet when $y = 16$ and $x^2 + z^2 = 16$ which is a circle of radius 4 sitting at $y = 16$.

2. Write down the equation of the cylinder with axis $x = 1$, $z = 2$ and with radius 3. [5 pt]

   Solution:

   The equation is $(x-1)^2 + (z-2)^2 = 9$.

3. Write down the function $f(x, y) = ...$ for the plane with intercepts $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 5)$. [5 pt]

   Solution:

   The plane has equation:

$$\frac{1}{2}x + \frac{1}{4}y + \frac{1}{5}z = 1$$

   So we rewrite as a function:

$$\begin{aligned}
\frac{1}{2}x + \frac{1}{4}y + \frac{1}{5}z &= 1 \\
\frac{5}{2}x + \frac{5}{4}y + z &= 5 \\
z &= 5 - \frac{5}{2}x - \frac{5}{4}y \\
f(x, y) &= 5 - \frac{5}{2}x - \frac{5}{4}y
\end{aligned}$$

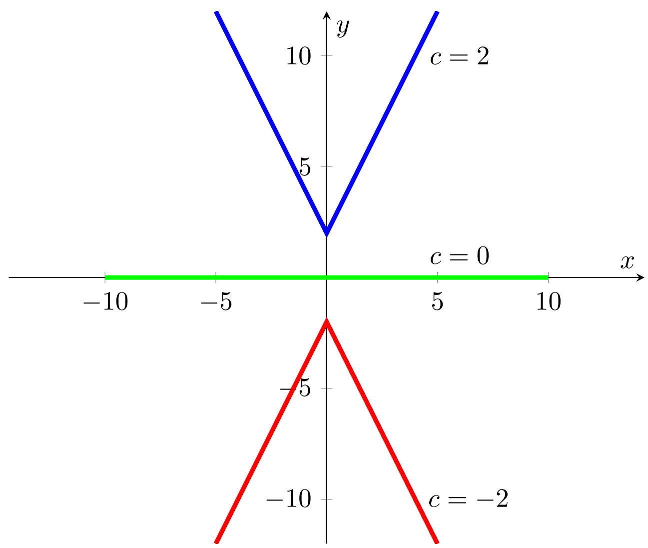
4. All together on one set of axes draw the level curves of $f(x,y) = \frac{y}{|x|+1}$ for the values $c = -2$, 0, and 2. Label each curve with its value of c. [10 pt]

Solution:

The level curves are:

- $c = 2$: $\frac{y}{|x|+1} = 2$ and so $y = -2(|x|+1) = -2|x| - 2$
- $c = 0$: $\frac{y}{|x|+1} = 0$ and so $y = 0(|x|+1) = 0$
- $c = -2$: $\frac{y}{|x|+1} = -2$ and so $y = 2(|x|+1) = 2|x| + 2$

Together:



5. Find the equation of the plane tangent to the graph of $f(x,y) = x^2y - y$ at the point where $x = 2$ and $y = -1$. Write it in the form $ax + by + cz = d$. [10 pt]

Solution:

We put $z = x^2y - y$ and rewrite as $x^2y - y - z = 0$. The gradient of the left side (call it g) is:

$$\nabla g = (2xy)\hat{\boldsymbol{\imath}} + (x^2 - 1)\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}}$$

At our point we get the normal vector:

$$\mathbf{n} = -4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}}$$

The point on the graph is:

$$(2, -1, f(2,-1)) = (2, -1, -3)$$

The plane then has equation:

$$\begin{aligned} -4(x-2) + 3(y+1) - 1(z+3) &= 0 \\ -4x + 8 + 3y + 3 - z - 3 &= 0 \\ -4x + 3y - z &= -8 \end{aligned}$$

6. Suppose the location of an object is given by $\mathbf{r}(t) = (t^2 + t)\hat{\boldsymbol{\imath}} + e^{2t-4}\hat{\boldsymbol{\jmath}}$. Suppose the temperature (in °C) of the plane (measured in meters) is given by $f(x, y) = x^2 + xy$

(a) Determine the instantaneous rate of change of temperature in °C/m that the object is experiencing as it passes through the point $(6, 1)$. [15 pt]

Solution:

We have $r'(t) = (2t + 1)\hat{\boldsymbol{\imath}} + 2e^{2t-4}\hat{\boldsymbol{\jmath}}$ and so $r'(2) = 5\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$. The corresponding unit vector is $\mathbf{u} = \frac{5}{\sqrt{29}}\hat{\boldsymbol{\imath}} + \frac{2}{\sqrt{29}}\hat{\boldsymbol{\jmath}}$. We have $f_x(x, y) = 2x + y$ and $f_y(x, y) = x$ and so $f_x(6, 1) = 13$ and $f_y(6, 1) = 6$ so the answer is the directional derivative:

$$D_{\mathbf{u}} f(6, 1) = \frac{5}{\sqrt{29}}(13) + \frac{2}{\sqrt{29}}(6)$$

(b) At the point $(6, 1)$ in which direction should it go if it wants to experience the maximum instantaneous change of temperature? [5 pt]

Solution:

If should go in the direction $\nabla f(6, 1)$. We have:

$$\nabla f(x, y) = (2x + y)\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$$

and so:

$$\nabla f(6, 1) = 13\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}$$

7. Find and categorize all three critical points (as relative maxima, relative minima, or saddle points) for the function $f(x, y) = 4xy - x^4 - y^4$. [20 pt]

Solution:

We have:

$$f_x = 4y - 4x^3 = 0$$
$$f_y = 4x - 4y^3 = 0$$

The first gives us $y = x^3$ which we then plug into the second to get

$$4x - 4(x^3)^3 = 0$$
$$x - x^9 = 0$$
$$x(1 - x^8) = 0$$

Thus we have $x = 0$ and $x = \pm 1$. Along with $y = x^3$ this gives us the three critical points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

The discriminant is then:

$$D(x, y) = (-12x^2)(-12y^2) - (4)^2$$

We check the points:

- $D(0, 0) = -16$ yields a saddle point.
- $D(1, 1) = 144 - 16 > 0$ and $f_{xx}(1, 1) = -$ yields a relative maximum.
- $D(-1, -1) = 144 - 16 > 0$ and $f_{xx}(1, 1) = -$ yields a relative maximum.

8. Use the method of Lagrange Multipliers to find the maximum and minimum values of the objective function $f(x, y) = xy$ subject to the constraint $x^2 + (y+1)^2 = 1$. [20 pt]

Solution:

We put $g(x, y) = 2x + (y+1)^2$ and write down our system:

$$\begin{aligned} y &= \lambda(2x) \\ x &= \lambda(2(y+1)) \\ x^2 + (y+1)^2 &= 1 \end{aligned}$$

If we multiply the first by $y+1$ and the second by x we get:

$$\begin{aligned} y(y+1) &= \lambda(2x)(y+1) \\ x^2 &= \lambda(2x)(y+1) \end{aligned}$$

Thus $x^2 = y^2 + y$ which we then plug into the third:

$$\begin{aligned} y^2 + y + y^2 + 2y + 1 &= 1 \\ 2y^2 + 3y &= 0 \\ y(2y+3) &= 0 \end{aligned}$$

Thus $y = 0$ or $y = -\frac{3}{2}$.
If $y = 0$ then we have $x^2 = 0$ and so $x = 0$ giving us $(0, 0)$.
If $y = -\frac{3}{2}$ then $x^2 = (-\frac{3}{2})^2 - \frac{3}{2} = \frac{9}{4} - \frac{6}{4} = \frac{3}{4}$ and so $x = \pm\frac{\sqrt{3}}{2}$ giving us $\left(\pm\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)$.

We then have:

- $f(0, 0) = 0$
- $f\left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = -\frac{3\sqrt{3}}{4}$ Minimum!
- $f\left(-\frac{\sqrt{3}}{2}, -\frac{3}{2}\right) = \frac{3\sqrt{3}}{4}$ Maximum!