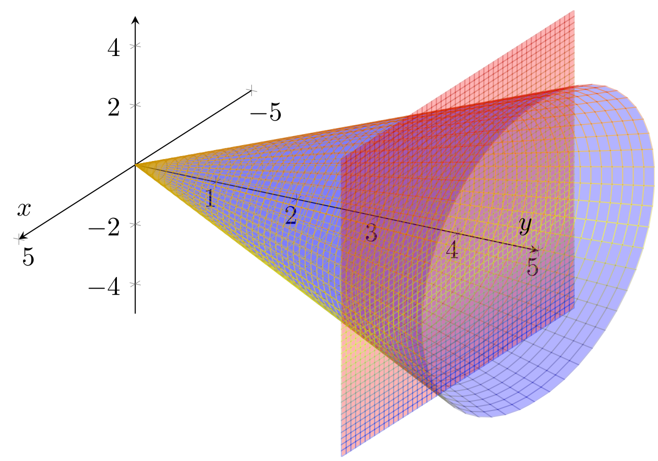
MATH241 Fall 2022 Exam 2 (Justin W-G) Solutions

1. Together in space, sketch the graphs of the two surfaces $y = \sqrt{x^2 + z^2}$ and $y = 4$. Describe the intersection of these two surfaces. Use both words and equations to be precise. [10 pt]

   Solution:

   The first is a cone with vertex at the origin and opening around the x-axis while the second is a plane at $y = 4$:

   The intersection is the circle $x^2 + z^2 = 16$ (radius 4) in the plane $y = 4$.

2. Write down the equation of the paraboloid with vertex at $(5, 0, 0)$ opening towards the negative x-axis. [5 pt]

   Solution:

   $x = 5 - y^2 - z^2$.

3. Write down the function $f(x, y) = ...$ for the plane with x-intercept $(2, 0, 0)$, z-intercept $(0, 0, 4)$, and no y-intercept. [5 pt]

   Solution:

   $f(x, y) = 4 - 2x$

4. All together on one set of axes draw the level curves of $f(x,y) = \frac{x}{|y|+1}$ for the values $c = -2$, 0, and 2. Label each curve with its value of c. [10 pt]

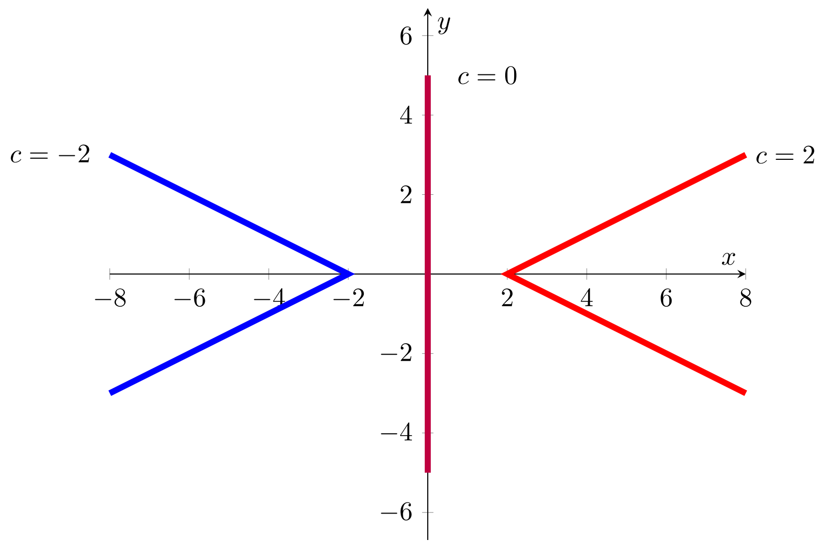
Solution:

The curves are:

$c = -2$: $\frac{x}{|y|+1} = -2$ so $x = -2|y| - 2$.

$c = 0$: $\frac{x}{|y|+1} = 0$ so $x = 0$.

$c = 2$: $\frac{x}{|y|+1} = 2$ so $x = 2|y| + 2$.



5. Use tangent plane approximation to find an approximation of: [10 pt]

$$\sin\left(\frac{11\pi}{12}\right)\sqrt{3.9}$$

Simplify as much as possible but don't approximate your final answer.

Solution:

Set $f(x,y) = \sin(x)\sqrt{y}$ and $(x_0, y_0) = (\pi, 4)$.

Then we have $f_x(x,y) = \cos(x)\sqrt{y}$ and $f_y = -\frac{\sin(x)}{2\sqrt{x}}$.

From here we have:

$$\begin{aligned}
f(x,y) &\approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \\
f\left(\frac{11\pi}{12}\right) &\approx f(\pi,4) + f_x(\pi,4)\left(\frac{11\pi}{12} - \pi\right) + f_y(\pi,4)(3.9 - 4) \\
&\approx \sin(\pi)\sqrt{4} + \cos(\pi)\sqrt{4}\left(-\frac{\pi}{12}\right) - \frac{\sin(\pi)}{2\sqrt{4}} \\
&\approx 0 + (-1)(2)\left(-\frac{\pi}{12}\right) - 0 \\
&\approx \pi/6
\end{aligned}$$

6. Suppose the location of an object is given by $\bar{r}(t) = (t^2+t)\hat{\imath} + e^{2t-4}\hat{\jmath}$. Suppose the temperature (in °C) of the plane (measured in meters) is given by $f(x,y) = x^2y - y$. At the instant when $t = 2$, is the object traveling in the direction of greatest temperature increase? Justify. [20 pt]

Solution:

First note that $\bar{r}(2) = 6\hat{\imath} + 1\hat{\jmath}$ so the object is at $(6,1)$.

Second we have $\bar{r}'(t) = (2t+1)\hat{\imath} + 2e^{2t-4}\hat{\jmath}$ and $r'(2) = 5\hat{\imath} + 2\hat{\jmath}$. This is the direction the object is moving.

Third we have $\nabla f(x,y) = 2xy\hat{\imath} + (x^2-1)\hat{\jmath}$ and $\nabla f(6,1) = 12\hat{\imath} + 35\hat{\jmath}$. This is the direction of greatest temperature increase.

We see that the direction of motion is not the same direction as the direction of greatest temperature increase.

7. Find and categorize both (there are two) critical points (as relative maxima, relative minima, or saddle points) for the function: [20 pt]

$$f(x,y) = \frac{1}{12}x^3 + 10xy + 5y^2$$

Solution:

We have:

$$\begin{aligned} f_x(x,y) &= \frac{1}{4}x^2 + 10y = 0 \\ f_y(x,y) &= 10x + 10y = 0 \end{aligned}$$

The second tells us $y = -x$. If we put this in the first we get:

$$\begin{aligned} \frac{1}{4}x^2 + 10(-x) &= 0 \\ x\left(\frac{1}{4}x - 10\right) &= 0 \end{aligned}$$

Thus $x = 0$ or $x = 40$ so the points are $(0,0)$ and $(40,-40)$.

Then $D(x,y) = \left(\frac{1}{2}x\right)(10) - (10)^2$ and so:

$(0,0)$: $D(0,0) = -$ so it's a saddle point.

$(40,-40)$: $D(40,-40) = +$ and $f_y(40,-40) = +$ so it's a relative minimum.

8. Find the absolute maximum and minimum of the function $f(x,y) = xy$ on the filled-in triangle with corners $(2,0)$, $(0,2)$, and $(2,2)$. [20 pt]

Solution:

For the critical points note that $f_x(x,y) = y$ and $f_x(x,y) = x$. If we set these equal to 0 we get the single critical point $(0,0)$. Note that $f(0,0) = 0$.

For the edge between $(2,0)$ and $(2,2)$: This is $x = 2$ so $f = 2y$. Since $0 \le y \le 2$ the minimum is 0 and the maximum is 4.

For the edge between $(0,2)$ and $(2,2)$: This is $y = 2$ so $f = 2x$. Since $0 \le x \le 2$ the minimum is 0 and the maximum is 4.

For the edge between $(2,0)$ and $(0,2)$: This is $y = 2-x$ so $f = x(2-x) = -x^2+2x$. Since $0 \le x \le 2$ we have to do a mini calc problem. Here f is a parabola with vertex $x = -b/(2a) = -2/2(-1) = 1$. $f(1) = 1$ and for the endpoints $f(0) = 0$ and $f(2) = 0$. Thus the minimum is 0 and the maximum is 1.

Overall the minimum is 0 and the maximum is 4.