MATH241 Fall 2023 Exam 2 (Justin W-G) Solutions

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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations unless otherwise specified.

1. Write TRUE or FALSE in the box to the right. No justification is required. Unreadable or ambiguous answers will be marked as incorrect. [10 pts]

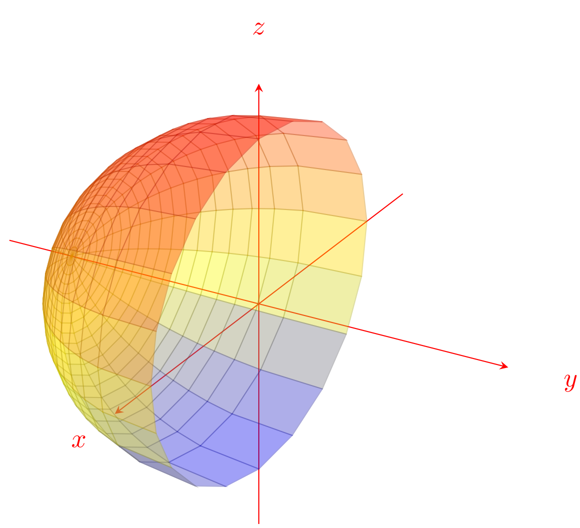
Solution:

Statement	TRUE/FALSE
$\nabla f(x,y)$ is perpendicular to the plane tangent to $f(x,y)$.	FALSE
For $D_{\bar{\boldsymbol{u}}}f$ the vector $\bar{\boldsymbol{u}}$ must be a unit vector.	TRUE
Lagrange Multipliers can find a maximum on a filled-in region.	FALSE
The discriminant is $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$.	TRUE
The equation $x^2 + y^2 + z^2 = 9$ is also a function.	FALSE

2. Sketch the graph of the equation $y = -\sqrt{9 - x^2 - z^2}$. Name the shape. [5 pts]

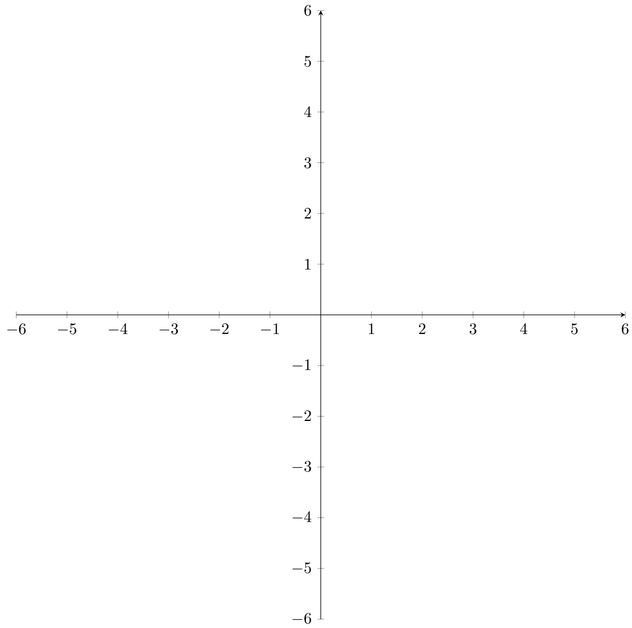
Solution:

This is a hemisphere:



3. All together on one xy-plane sketch the level curves for $f(x, y) = x - y^2$ for $c = -2, 0, 2$. Label each with its value of c. [10 pts]

Solution:



4. Suppose $f(x, y) = xy + 3y^2$. If $\bar{\boldsymbol{u}}$ is a unit vector which makes an angle of $\pi/6$ with ∇f at $(2, -1)$, find $D_{\bar{\boldsymbol{u}}} f(2, -1)$. [10 pts]

Solution:

We know that:

$$D_{\bar{\boldsymbol{u}}} f = \bar{\boldsymbol{u}} \cdot \nabla f = ||\bar{\boldsymbol{u}}|| \, ||\nabla f|| \cos\theta = ||\nabla f|| \cos\theta$$

We have $\nabla f(x, y) = y \,\hat{\boldsymbol{\imath}} + (x + 6y) \,\hat{\boldsymbol{\jmath}}$.
Hence at $(2, -1)$ with angle $\theta = \pi/6$ we have:

$$D_{\bar{\boldsymbol{u}}} f(2, -1) = || - 1 \,\hat{\boldsymbol{\imath}} - 4 \,\hat{\boldsymbol{\jmath}}|| \cos(\pi/6) = \sqrt{17}(\sqrt{3}/2)$$

5. Find a vector perpendicular to the graph of $f(x,y) = x\sin(xy)$ at the point $(2, \pi/12)$. Evaluate the trigonometry but don't simplify further. [7 pts]

Solution:

We write this as $z = x\sin(xy)$ and then as $x\sin(xy) - z = 0$. We then set $g(x,y,z) = x\sin(xy) - z$ because then the graph of f is the level surface of g. Then we find:

$$\begin{aligned}
\nabla g(x,y,z) &= (\sin(xy) + xy\cos(xy))\,\hat{\boldsymbol{\imath}} + x^2\cos(xy)\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\
\nabla g(2,\pi/12,z) &= (\sin(\pi/6) + (\pi/6)\cos(\pi/6))\,\hat{\boldsymbol{\imath}} + 4\cos(\pi/6)\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\
&= (1/2 + (\pi/6)(\sqrt{3}/2))\,\hat{\boldsymbol{\imath}} + 4(\sqrt{3}/2)\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}}
\end{aligned}$$

6. The temperature of the plane in $°C$ at (x, y) is $T(x, y) = x\sqrt{y^2 + 9}$. An object is following the path given by the following where t is in seconds: [10 pts]

$$\bar{\boldsymbol{r}}(t) = 3t\,\hat{\boldsymbol{\imath}} + t^2\,\hat{\boldsymbol{\jmath}}$$

What is the rate of change in $°C/s$ that the object is undergoing at the instant when $t = 2$?

Solution:

The chain rule gives us:

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} \\ &= \sqrt{y^2 + 9}(3) + \frac{1}{2}(y^2 + 9)^{-1/2}(2y)(2t) \\ &= \sqrt{t^4 + 9}(3) + \frac{1}{2}(t^4 + 9)^{-1/2}(2t^2)(2t)\end{aligned}$$

At $t = 2$ we then have:

$$\left.\frac{dT}{dt}\right|_{t=2} = \sqrt{25}(3) + \frac{1}{2}(25)^{-1/2}(8)(4) = 15 + \frac{1}{10}(32)$$

7. Use tangent plane approximation to approximate the value of $\ln(2x+y)$ at $(x,y) = (-0.9, 3.2)$. [8 pts] Simplify.

Solution:

We define $f(x,y) = \ln(2x+y)$ and use $(x_0, y_0) = (-1, 3)$. We have:

$$\begin{aligned} f(-1,3) &= \ln(1) = 0 \\ f_x(x,y) &= \frac{2}{2x+y} && \text{so } f_x(-1,3) = 2 \\ f_y(x,y) &= \frac{1}{2x+y} && \text{so } f_y(-1,3) = 1 \end{aligned}$$

Thus we have:

$$\begin{aligned} f(-0.9, 3.2) &\approx f(-1,3) + f_x(-1,3)(-0.9-(-1)) + f_y(-1,3)(3.2-3) \\ &\approx 0 + 2(0.1) + 1(0.2) \\ &\approx 0.4 \end{aligned}$$

8. Find all three of the critical points for the function $h(x,y) = x^2y - 2x^2 - y^2$. For each critical point calculate if it is a relative maximum, relative minimum, or saddle point. [20 pts]

Solution:

We find the partials and set them equal to 0:

$$h_x(x,y) = 2xy - 4x = 0$$
$$h_y(x,y) = x^2 - 2y = 0$$

The first factors to $2x(y-2) = 0$ so $x = 0$ or $y = 2$.
If $x = 0$ then the second yields $2y = 0$ so we get $(0,0)$.
If $y = 2$ then the second yields $x^2 = 4$ so we get $(\pm 2, 2)$.
We then find the discriminant:

$$D(x,y) = (2y-4)(-2) - (2x)^2$$

Then we test our points:

- $D(0,0) = 8$ so then since $h_{yy}(0,0) = -$ this is a relative maximum.
- $D(2,2) = -$ so this is a saddle point.
- $D(-2,2) = -$ so this is a saddle point.

9. Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y) = 4xy$ on the circle $x^2 + (y-1)^2 = 1$. [20 pts]

Solution:

We have $g(x, y) = x^2 + (y-1)^2$. We set up the system:

$$\begin{aligned} 4y &= \lambda(2x) \\ 4x &= \lambda(2(y-1)) \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

Solving for λ in the first two and setting them equal yields:

$$\begin{aligned} \frac{4y}{2x} &= \frac{4x}{2(y-1)} \\ \frac{y}{x} &= \frac{x}{y-1} \\ x^2 &= y^2 - y \end{aligned}$$

If we plug this into the constraint:

$$\begin{aligned} y^2 - y + (y-1)^2 &= 1 \\ y^2 - y + y^2 - 2y + 1 &= 1 \\ 2y^2 - 3y &= 0 \\ y(2y-3) &= 0 \end{aligned}$$

Thus $y = 0$ or $y = 3/2$.
If $y = 0$ then $x^2 + (0-1)^2 = 1$ yields $x = 0$ and we have $(0, 0)$.
If $y = 3/2$ then $x^2 + (3/2-1)^2 = 1$ gives us $x^2 = 3/4$ so $x = \pm\sqrt{3}/2$ and we have $\left(\pm\sqrt{3}/2, 3/2\right)$.
If we test these we find:

- $f(0, 0) = 4(0)(0)$
- $f(\sqrt{3}/2, 3/2) = 4(\sqrt{3}/2)(3/2) = 3\sqrt{3}$
- $f(-\sqrt{3}/2, 3/2) = 4(-\sqrt{3}/2)(3/2) = -3\sqrt{3}$

Thus the maximum is $3\sqrt{3}$ and the minimum is $-3\sqrt{3}$.