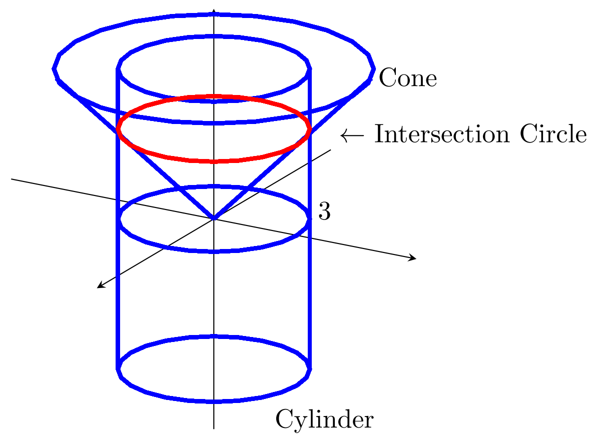
**Math 241 Exam 2 ~~~~~~ 2020 Solutions** **ĵustîn Wyss-Gallîfent**

1. (a) Sketch the surfaces $x^2 + y^2 = 9$ and $z = \sqrt{x^2 + y^2}$ together. Name the surfaces and describe what the intersection is. [12 pts]

**Solution:**




(b) Write down the equation of the upper hemisphere with center $(2, 3, 0)$ and radius 5. [8 pts]

**Solution:**

We have:

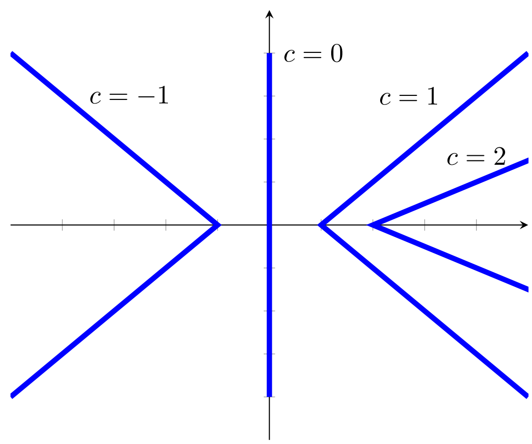
$$z = \sqrt{25 - (x-2)^2 - (y-3)^2}$$

2. (a) All together on one set of axes sketch the level curves for $f(x,y) = \frac{x}{|y|+1}$ for $c = -1, 0, 1, 2$. [10 pts] Label each curve with its value of $c$.

**Solution:** The level curves are:

$$\begin{array}{lll} \frac{x}{|y|+1} = -1 & \rightarrow & x = -|y| - 1 \\ \frac{x}{|y|+1} = 0 & \rightarrow & x = 0 \\ \frac{x}{|y|+1} = 1 & \rightarrow & x = |y| + 1 \\ \frac{x}{|y|+1} = 2 & \rightarrow & x = 2|y| + 2 \end{array}$$

The sketches are:




(b) Find the symmetric equation of the line in the plane $y = 2$ which is tangent to the intersection of the plane with the function $f(x,y) = 3x^2 + y^2$ at the point $(1, 2, 7)$. [10 pts]

**Solution:** We have the point $(1, 2, 7)$, all we need is a direction vector. Since we are in the plane $y = 2$ observe that $f_x(x,y) = 6x$ and so $f_x(1,2) = 6$ which means that instantaneously if $x$ increases by 1 then $z$ increases by 6. Thus we have $\mathbf{L} = 1\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$ and so the symmetric equations are:

$$\frac{x-1}{1} = \frac{z-7}{6},\ y = 2$$

3. (a) Use tangent plane approximation at $(4,27)$ to approximate the value of $\sqrt{3.9}+\sqrt[3]{28}$. [6 pts]

**Solution:**

If we put $f(x,y)=\sqrt{x}+\sqrt[3]{y}$. then we want $f(3.9,28)$. Observe that:

$$\begin{aligned} f(4,27) &= 2+3=5 \\ f_x(x,y) &= \frac{1}{2}x^{-1/2} \\ f_x(4,27) &= \frac{1}{4} \\ f_y(x,y) &= \frac{1}{3}y^{-2/3} \\ f_y(4,27) &= \frac{1}{27} \end{aligned}$$

Therefore:

$$f(3.9,28)\approx 5+\frac{1}{4}(3.9-4)+\frac{1}{27}(28-27)$$

(b) Suppose the temperature at $(x,y)$ is given by $f(x,y)=xy+x^2y$. If an object is following the curve $\mathbf{r}(t)=t^2\hat{\boldsymbol{\imath}}+\left(t^3-t\right)\hat{\boldsymbol{\jmath}}$, what instantaneous temperature change is the object experiencing with respect to distance at the instant $t=2$? [14 pts]

**Solution:** At $t=2$ the object is at $\mathbf{r}(2)=4\hat{\imath}+6\hat{\jmath}$ or $(4,6)$. Since $\mathbf{r}'(t)=2t\hat{\boldsymbol{\imath}}+(3t^2-1)\hat{\boldsymbol{k}}$ the direction it is going is $\mathbf{r}'(2)=4\hat{\boldsymbol{\imath}}+11\hat{\boldsymbol{\jmath}}$. So we want the directional derivative of $f$ in this direction at $(4,6)$. First find the unit vector:

$$\mathbf{u}=\frac{r'(2)}{||\mathbf{r}'(2)||}=\frac{4}{137}\hat{\boldsymbol{\imath}}+\frac{11}{137}\hat{\boldsymbol{\jmath}}$$

Since we have:

$$\begin{aligned} f_x(x,y) &= y+2xy \\ f_y(x,y) &= x+x^2 \end{aligned}$$

We have:

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= \frac{4}{137}(y+2xy)+\frac{11}{137}(x+x^2) \\ D_{\mathbf{u}}f(4,6) &= \frac{4}{137}(54)+\frac{11}{137}(20) \end{aligned}$$

4. Find all three (guaranteed to be three!) of the critical points for the function: [20 pts]

$$f(x, y) = x^2y - xy + 3y^2$$

For each critical point calculate if it is a relative maximum, relative minimum or saddle point.

**Solution:** We solve the system:

$$f_x = 2xy - y = 0 \qquad (1)$$
$$f_y = x^2 - x + 6y = 0 \qquad (2)$$

From (1) we get $y(2x - 1) = 0$ so $y = 0$ or $x = \frac{1}{2}$.

If $y = 0$ then (2) gives us $x^2 - x = 0$ or $x(x - 1) = 0$ so $x = 0$ or $x = 1$ and we have two points $(0, 0)$ and $(1, 0)$.

If $x = \frac{1}{2}$ then (2) gives us $\frac{1}{4} - \frac{1}{2} + 6y = 0$ so $y = \frac{1}{24}$ and we have one point $\left(\frac{1}{2}, \frac{1}{24}\right)$.

We find the discriminant:

$$D(x, y) = (2y)(6) - (2x - 1)^2$$

Then we check the points:

- $(0, 0)$: We have $D(0, 0) = -$ so $(0, 0)$ is a saddle.
- $(1, 0)$: We have $D(1, 0) = -$ so $(0, 0)$ is a saddle.
- $(1/2, 1/24)$: We have $D(1/2, 1/24) = +$ and then $f_y(1/2, 1/24) = +$ so $(1/2, 1/24)$ is a relative minimum.

5. Use the method of Lagrange Multipliers to find the maximum and minimum of the function $f(x, y) = xy - 2y$ subject to the constraint $x^2 + 4y^2 = 4$. [20 pts]

**Solution:**

We solve the system:

$$y = \lambda(2x) \tag{3}$$
$$x - 2 = \lambda(8y) \tag{4}$$
$$x^2 + 4y^2 = 4 \tag{5}$$

We can rewrite (3) as $(x - 2)y = \lambda(2x)(x - 2)$ and (4) as $(x - 2)y = \lambda(8y)(y)$ and so we have:

$$\lambda(2x)(x - 2) = \lambda(8y)(y)$$
$$\lambda(x^2 - 2x - 4y^2) = 0$$

Thus either $\lambda = 0$ or $x^2 - 2x - 4y^2 = 0$.

- If $\lambda = 0$ then (3) tells us $y = 0$ and (4) tells us $x = 2$ and $(2, 0)$ satisfies (5) so this is one point.
- If $x^2 - 2x - 4y^2 = 0$ then $4y^2 = x^2 - 2x$ then (5) tells us $x^2 + x^2 - 2x = 4$ which can be rewritten as $x^2 - x - 2 = 0$ or $(x - 2)(x + 1) = 0$ so $x = 2$ or $x = -1$.
  - If $x = 2$ then (5) tells us that $y = 0$ so we get $(2, 0)$ again.
  - If $x = -1$ then (5) tells us that $y = \pm\sqrt{3}/2$ so we get $(-1, \pm\sqrt{3}/2)$.

We then check these points:

- $(2, 0)$: We have $f(2, 0) = 0$.
- $(-1, \sqrt{3}/2)$: We have $f(-1, \sqrt{3}/2) = -3\sqrt{3}/2$. Minimum.
- $(-1, -\sqrt{3}/2)$: We have $f(-1, -\sqrt{3}/2) = 3\sqrt{3}/2$. Maximum.