MATH 241 Sections Exam 2 Spring 2021 (JWG)

Solutions

Exam Submission:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute.
2. Tag your problems! Please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use your notes and the textbook. You are permitted to use a calculator for basic arithmetic.
3. You are not permitted to use other resources. Thus no friends, internet, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Work Shown:

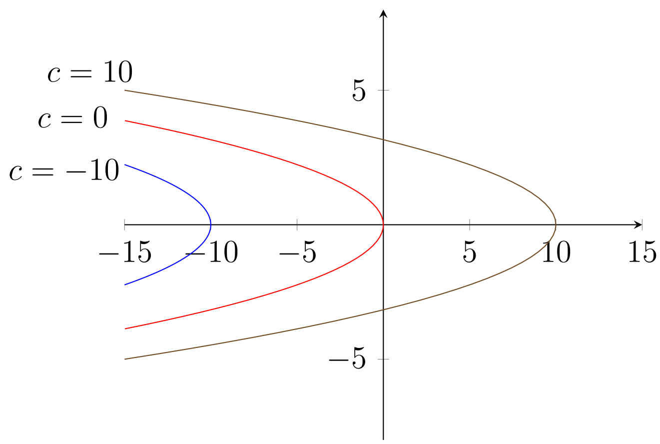
1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. (a) On the same set of axes plot the level curves for $f(x, y) = x + y^2$ for the values $c = -10, 0, 10$ and label each curve with its value of c. Also, label your axes! [10 pts]

Solution:

The three curves are:

$$x + y^2 = -10 \Rightarrow x = -10 - y^2$$
$$x + y^2 = 0 \Rightarrow x = -y^2$$
$$x + y^2 = 10 \Rightarrow x = 10 - y^2$$



(b) Sketch the level surface for $f(x, y, z) = y + 10 - \sqrt{x^2 + z^2}$ for the value $c = 5$. Also, label your axes! [5 pts]

Solution:

When $c = 5$ we get:

$$y + 10 - \sqrt{x^2 + z^2} = 5$$
$$y = -5 + \sqrt{x^2 + z^2}$$

I'm not going to typset the picture here (takes a while!) but this is a cone with vertex at $(0, -5, 0)$ which opens in the positive y-direction.

2. Find the equation of the plane tangent to the graph of $f(x, y) = x^2y + x$ at the point where $x = 2$ and $y = 3$. Simplify this to the form $ax + by + cz = d$. [10 pts]

Solution:

The point we're interested in is: $(2, 3, f(2, 3)) = (2, 3, 14)$.

If we rewrite the function as $z = x^2y + x$ and then $x^2y + x - z = 0$ and then we let $g(x, y, z) = x^2y + x - z$ then the gradient of g is perpendicular to the graph and can be used as the normal vector. We have:

$$\begin{aligned} \nabla g(x, y, z) &= (2xy + 1)\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \\ \nabla g(2, 3, 14) &= 13\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}} - 1\,\hat{\boldsymbol{k}} \end{aligned}$$

Thus the plane is:

$$\begin{aligned} 13(x - 2) + 4(y - 3) - 1(z - 14) &= 0 \\ 13x + 4y - z &= 24 \end{aligned}$$

3. Suppose for some $f(x, y)$ (not given) and for some unit vector $\bar{\boldsymbol{u}}$ (also not given) you calculate that $D_{\bar{\mathbf{u}}} f(3, 4) = 8$ and $\nabla f(3, 4) = 5\,\hat{\boldsymbol{\imath}} + 5\,\hat{\boldsymbol{\jmath}}$. Explain how you know you made a mistake. [5 pts]

Solution:

Since the maximum directional derivative is $||\nabla f(3, 4)|| = ||5\,\hat{\boldsymbol{\imath}} + 5\,\hat{\boldsymbol{\jmath}}|| = \sqrt{50}$ and $8 > \sqrt{50}$ we have a problem

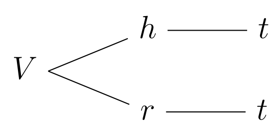
4. A cylinder is growing such that its radius is changing at 2 cm/sec while its volume is changing at $3\pi\,\text{cm}^3/\text{sec}$. At what rate is its height changing at the instant when the height is 100 cm and the volume is $20\pi\,\text{cm}^3$? Include units. [10 pts]

Solution:

We know that the formula for the volume of a cylinder is:

$$V = \pi r^2 h$$

Hence we have the following tree diagram:



By the chain rule then:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}\\ 3\pi &= (2\pi r h)(2) + (\pi r^2)\frac{dh}{dt}\end{aligned}$$

To figure out $\frac{dh}{dt}$ we need r and h. When $h = 100$ and $V = 20\pi$ we have $20\pi = \pi r^2(100)$ and so $r = \sqrt{0.2}$ and so then:

$$\begin{aligned}3\pi &= \left(2\pi\sqrt{0.2}(100)\right)(2) + \pi(0.2)\frac{dh}{dt}\\ 3 &= 400\sqrt{0.2} + 0.2\frac{dh}{dt}\\ 0.2\frac{dh}{dt} &= 3 - 400\sqrt{0.2}\\ \frac{dh}{dt} &= 15 - 2000\sqrt{0.2}\end{aligned}$$

5. An object in the plane follows the path $\bar{\boldsymbol{r}}(t) = t^2\,\hat{\boldsymbol{\imath}} + (2t^2 - t)\,\hat{\boldsymbol{\jmath}}$. The temperature at the point (x, y) is given by $f(x, y) = x^2 + x^2y^2$. Assume temperature in $^\circ C$ and distance is in meters and give units.

   (a) At $t = 3$ what is the direction of maximum increase of f? [5 pts]

   Solution:

   When $t = 3$ the object is located at $\bar{\boldsymbol{r}}(3) = 9\,\hat{\boldsymbol{\imath}} + 15\,\hat{\boldsymbol{\jmath}}$ or $(9, 15)$. We have direction of maximum increase:

$$\nabla f(x, y) = (2x + 2xy^2)\,\hat{\boldsymbol{\imath}} + 2x^2y\,\hat{\boldsymbol{\jmath}}$$
$$\nabla f(9, 15) = 4068\,\hat{\boldsymbol{\imath}} + 2430\,\hat{\boldsymbol{\jmath}}$$

   (b) At $t = 3$ what instantaneous temperature change is the object experiencing? [10 pts]

   Solution:

   We have $\bar{\boldsymbol{v}}(t) = 2t\,\hat{\boldsymbol{\imath}} + (4t - 1)\,\hat{\boldsymbol{\jmath}}$ and so $\bar{v}(3) = 6\,\hat{\boldsymbol{\imath}} + 11\,\hat{\boldsymbol{\jmath}}$. We need the directional derivative at $(9, 15)$ in that direction as a unit vector. The corresponding unit vector is:

$$\bar{\boldsymbol{u}} = \frac{6}{\sqrt{157}}\,\hat{\boldsymbol{\imath}} + \frac{11}{\sqrt{157}}\,\hat{\boldsymbol{\jmath}}$$

   Then the directional derivative is:

$$D_{\bar{\boldsymbol{u}}} f(9, 15) = \frac{6}{\sqrt{157}}(4068) + \frac{11}{\sqrt{157}}(2430)$$

6. **Note:** Let G be the largest single digit of your UID. Write down your UID and the value of G and mark them clearly. In the problem below, replace G by the appropriate value before proceeding. [15 pts]

Consider the function:

$$f(x,y) = x^2y + 2y^2 - (2G+2)xy + (G^2+2G-15)y$$

This function has three critical points. Find and classify as either relative max, relative min, or saddle points.

Solution:

For a general G (your particular solutions will be far less messy but you can perhaps use this to check your algebra) we have:

$$\begin{aligned} f_x(x,y) &= 2xy - (2G+2)y \\ f_y(x,y) &= x^2 + 4y - (2G+2)x + G^2 + 2G - 15 \end{aligned}$$

If we set the first equal to 0 and solve we get $y(2x-(2G+2)) = 0$ so either $y = 0$ or $x = G+1$.

If $y = 0$ then the second yields:

$$\begin{aligned} x^2 - (2G+2)x + G^2 + 2G - 15 &= 0 \\ (x-(G+5))(x-(G-3)) &= 0 \end{aligned}$$

So that either $x = G+5$ or $x = G-3$, hence giving critical points $(G+5,0)$ and $(G-3,0)$.

If $x = G+1$ the the second yields:

$$\begin{aligned} (G+1)^2 + 4y - (2G+2)(G+1) + G^2 + 2G - 15 &= 0 \\ G^2 + 2G + 1 + 4y - 2G^2 - 4G - 2 + g^2 + 2G - 15 &= 0 \\ 4y &= 16 \\ y &= 4 \end{aligned}$$

Hence we also have critical point $(G+1,4)$.

We find $D(x,y) = (2y)(4) - (2x-(2G+2))^2$ and we check:

- $D(G-3,0) = -$ so saddle point.
- $D(G+5,0) = -$ so saddle point.
- $D(G+1,4) = 32 - (2(G+1)-(2G+2))^2 = 32 = +$ and $f_{xx}(G+1,4) = 8 = +$ so relative minimum.

7. **Note:** Let E be the sum of the leftmost three digits of your UID. Write down your UID and the value of E and mark them clearly. In the problem below, replace E by the appropriate value before proceeding. [15 pts]

Find the maximum of the function:

$$f(x, y) = Exy$$

Where (x, y) is constrained within the region in the plane between $y = x^2$ and $y = 9$.

Solution:

We have $f_x = Ey$ and $f_y = Ex$ so the single critical point is $(0, 0)$. Then $f(0, 0) = 0$.

The bottom edge is the parabola and is $y = x^2$ for $-3 \leq x \leq 3$. On that edge we have $f(x, y) = Exy = Ex(x^3) = Ex^3$. This has a minimum of $E(-3)^3 = -27E$ when $x = -3$ and this has a maximum of $E(3)^3 = 27E$ when $x = 3$.

The top edge is the line segment and is $y = 9$ for $-3 \leq x \leq 3$. On that edge we have $f(x, y) = Exy = 9Ex$. This has a minimum of $9E(-3) = -27E$ when $x = -3$ and this has a maximum of $9E(3) = 27E$ when $x = 3$.

Thus the maximum is 27.

(I only realized when writing the solutions that I asked for only the maximum.)

8. **Note:** Let F be the sum of the leftmost four digits of your UID. Write down your UID and the value of F and mark them clearly. In the problem below, replace F by the appropriate value before proceeding. [15 pts]

Use Lagrange Multipliers to find the minimum value of the function:

$$f(x, y) = (x - F)^2 + y^2$$

Subject to the constraint:

$$x^2 + y^2 = F^2$$

Solution:

The constraint function is $G(x, y) = x^2 + y^2$. The system we need to solve is:

$$2(x - F) = \lambda 2x \quad (1)$$
$$2y = \lambda 2y \quad (2)$$
$$x^2 + y^2 = F^2 \quad (3)$$

From (1) we get $2(x - F)y = \lambda 2xy$ and from (2) we get $2xy = \lambda 2xy$. If we set these equal we get $2(x - F)y = 2xy$ so either $y = 0$ or $x - F = x$. The latter is not possible and so $y = 0$ and then from the constraint we get $x = \pm F$ giving us points $(\pm F, 0)$.

Then we have:

- $f(+F, 0) = 0$
- $f(-F, 0) = 4F^2$

Thus the minimum is 0 and the maximum is $4F^2$.