MATH241 Exam 2 Spring 2022 (Justin Wyss-Gallifent)

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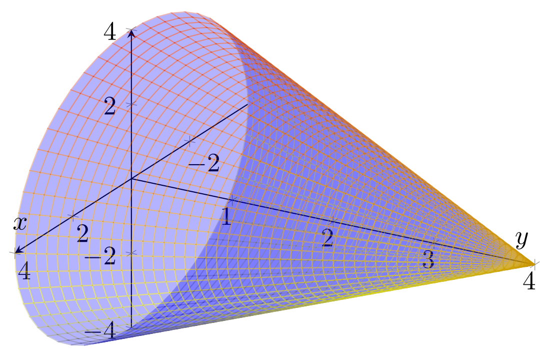
Directions: Please do work in the spaces provided and do not spill over to other pages - the exams will be scanned into Gradescope and auto-tagged this way. Use only methods taught in this course and show work as indicated. No calculators or other devices permitted. Numerical answers do not need to be simplified. Good luck!

1. Draw a reasonable sketch of the equation: [10 pts]

$$y = 4 - \sqrt{x^2 + z^2}$$

Solution:

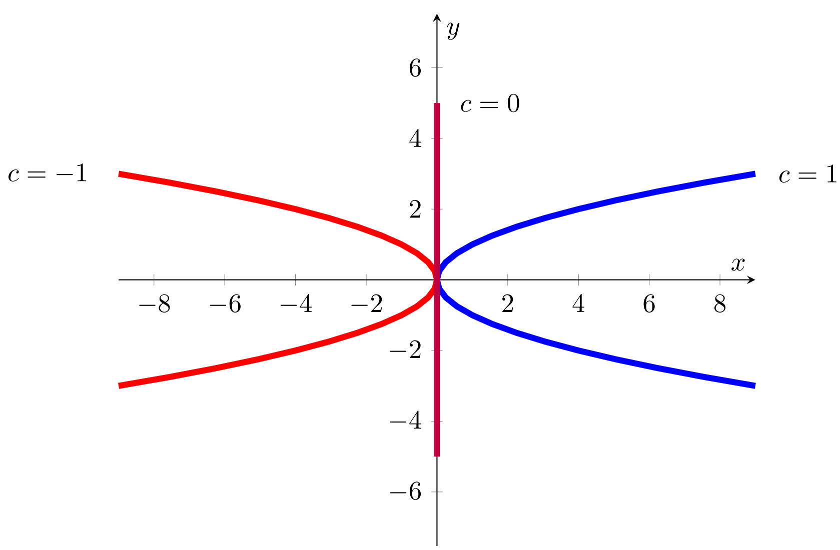
I don't expect anything this fancy but it's a cone with vertex at $(0, 4, 0)$ opening in the negative y direction.



2. All together on one set of 2D axes draw the level curves of $f(x, y) = \frac{x}{y^2}$ for the values $c = -1$, 0, and 1. Label each curve with its value of c. [10 pt]

Solution:

The level curves are $\frac{x}{y^2} = 1$, or $x = y^2$, $\frac{x}{y^2} = 0$, or $x = 0$, and $\frac{x}{y^2} = 1$, or $x = y^2$.



3. An object is traveling along the line $y = 2x+1$ heading up and to the right. If the temperature at (x, y) in degrees celsius is given by $f(x, y) = x^2y + x - y$, and if the plane is measured in meters, what is the instantaneous temperature change the object is experiencing at the instant when $x = 3$? [15 pts]

Solution:

The direction is $1\hat{\imath} + 2\hat{\jmath}$ and as a unit vector this is $\boldsymbol{u} = \frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{\jmath}$. Then $f_x = 2xy + 1$ and $f_y = x^2 - 1$ and the point is $(3, 7)$ and so:

$$D_{\boldsymbol{u}}f(3,7) = \frac{1}{\sqrt{5}}(2(3)(7)+1) + \frac{2}{\sqrt{5}}(3^2-1)$$

4. Suppose all you know is: [10 pts]

$$\begin{aligned} z &= x\sin(xy) \\ x &= e^{st} \\ y &= st \end{aligned}$$

Use the Chain rule (from 13.4) to find $\frac{\partial z}{\partial s}$ at $s = 2$ and $t = 1$.

Solution:

We have:

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (\sin(xy) + xy\cos(xy))\left(te^{st}\right) + \left(x^2\cos(xy)\right)(t) \\ &= \left(\sin(ste^{st}) + ste^{st}\cos(ste^{st})\right)\left(te^{st}\right) + \left(e^{2st}\cos(ste^{st})\right)(t) \end{aligned}$$

At $t = 1$ and $s = 2$ we get:

$$\left(\sin(2e^2) + 2e^2\cos(2e^2)\right)\left(e^2\right) + \left(e^4\cos(2e^2)\right)$$

5. Find the parameterization $\boldsymbol{r}(t) = ...$ of the line perpendicular to the surface $z = x^2 + y$ at the point $(1, 2, 3)$. [10 pts]

Solution:

We put $g(x, y, z) = z - x^2 - y$ and find $\nabla g = -2x\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$ and then $\nabla g(1, 2, 3) = -2\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$ and so the parameterization is $\boldsymbol{r}(t) = (-2t + 1)\hat{\boldsymbol{\imath}} + (-1t + 2)\hat{\boldsymbol{\jmath}} + (1t + 3)\hat{\boldsymbol{k}}$.

6. Use tangent plane approximation with a nearby reasonable point to approximate the value of: [10 pts]

$$(\ln(e - 0.1))\sqrt{9.2}$$

Solution:

Put $f(x, y) = \ln(x)\sqrt{y}$, then we want $f(e - 0.1, 9.2)$. Using an anchor point of $(e, 9)$ we have:

$$\begin{aligned} f(e, 9) &= \ln(e)\sqrt{9} = 3 & & \\ f_x(x, y) &= \frac{1}{x}\sqrt{y} & f_x(e, 9) &= \frac{1}{e}\sqrt{9} = \frac{9}{e} \\ f_y(x, y) &= \ln(x)\frac{1}{2\sqrt{y}} & f_y(e, 9) &= \ln(e)\frac{1}{2\sqrt{9}} = \frac{1}{6} \end{aligned}$$

Thus we have:

$$\begin{aligned} f(e - 0.1, 9.2) &\approx f(e, 9) + f_x(e, 9)(e - 0.1 - e) + f_y(e, 9)(9.2 - 9) \\ &\approx 3 + \frac{9}{e}(-0.1) + \frac{1}{6}(0.2) \end{aligned}$$

7. Find and categorize both (there are two) critical points (as relative maxima, relative minima, or saddle points) for the function: [15 pt]

$$f(x, y) = \frac{1}{12}x^3 + 10xy + 5y^2$$

Solution:

We solve:

$$f_x = \frac{1}{4}x^2 + 10y = 0$$
$$f_y = 10x + 10y = 0$$

The second tells us that $x = -y$ so we plug this into the first:

$$\frac{1}{4}(-y)^2 + 10y = 0$$
$$y^2 + 40y = 0$$
$$y(y + 40) = 0$$

When $y = 0$ we get $x = 0$ and when $y = -40$ we get $x = 40$ so we have two critical points $(0, 0)$ and $(40, -40)$.

We then have:

$$D(x, y) = \left(\frac{1}{2}x\right)(10) - (10)^2$$

Then $D(0, 0) = -100$ so $(0, 0)$ is a saddle point.

And $D(40, -40) = 100$ and $f_x(40, -40) = +$ so $(40, -40)$ is a relative minimum.

8. Use the method of Lagrange Multipliers to find the maximum and minimum values of the objective function $f(x, y) = 2xy$ subject to the constraint $(x-2)^2 + y^2 = 4$. [20 pt]

Solution:

We define $g(x, y) = (x-2)^2 + y^2$ and then our system is:

$$\begin{aligned} 2y &= \lambda \cdot 2(x-2) \\ 2x &= \lambda \cdot 2y \\ (x-2)^2 + y^2 &= 4 \end{aligned}$$

There are a variety of ways to solve this. If we multiply the first by y and the second by $x - 2$ then they become:

$$\begin{aligned} 2y^2 &= \lambda \cdot 2(x-2)y \\ 2x(x-2) &= \lambda \cdot 2(x-2)y \end{aligned}$$

Thus the left sides are equal and so $y^2 = x^2 - 2x$. If we plut this into the third one:

$$\begin{aligned} (x-2)^2 + x^2 - 2x &= 4 \\ x^2 - 4x + 4 + x^2 - 2x &= 4 \\ 2x^2 - 6x &= 0 \\ 2x(x-3) &= 0 \end{aligned}$$

If $x = 0$ then $y^2 = x^2 - 2x = 0$ so $y = 0$ yielding $(0, 0)$.

If $x = 3$ then $y^2 = x^2 - 2x = 3$ so $y = \pm\sqrt{3}$ yielding $(3, \sqrt{3})$ and $(3, -\sqrt{3})$.

We then test:

$f(0, 0) = 0$

$f(3, \sqrt{3}) = 6\sqrt{3}$ the maximum!

$f(3, -\sqrt{3}) = -6\sqrt{3}$ the minimum!