MATH241 Spring 2023 Exam 2 Solutions (Justin W-G)

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Instructions:

1. Please do all problems on the pages and in the spaces provided. This exam will be scanned into Gradescope and if your answers are not in the correct locations they will not be found or graded!
2. Only simplify Calculus 3 related calculations.

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1. Write T for True or F for False in the box to the right. No justification is required. Unreadable or ambiguous letters will be marked as incorrect. [10 pts]

Solution:

Statement	T or F
The graphs of $z = x^2 + y^2$ and $z = 5$ meet.	T
f_{xx} always equals f_{yy}.	F
$D_{\bar{\mathbf{u}}} f = \bar{\boldsymbol{u}} \cdot \nabla f$	T
$f_x = D_{\bar{\mathbf{u}}} f$ for some $\bar{\boldsymbol{u}}$	T
Tangent plane approximations are usually exactly correct.	F

2. Sketch the graph of the equation $y = -\sqrt{9 - x^2 - z^2}$. Include some sense of size and position. Name the shape. [10 pts]

Solution:

3. Write down the equation for a cone with vertex at $(2, 0, 0)$ opening toward/around the negative x-axis. [5 pts]

Solution:

$x = 2 - \sqrt{y^2 + z^2}$

4. Find a vector perpendicular to the graph of $f(x, y) = x\cos(xy)$ at the point $(2, \pi/12)$. [8 pts]

Solution:

Put:

$$\begin{aligned} z &= x\cos(xy) \\ x\cos(xy) - z &= 0 \end{aligned}$$

We assign $g(x, y, z) = x\cos(xy) - z$ and take the gradient:

$$\nabla g = (\cos(xy) - xy\sin(xy))\hat{\boldsymbol{\imath}} - x^2\sin(xy)\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}}$$

At the point $(2, pi/12)$:

$$\begin{aligned} \nabla g &= (\cos(pi/6) - (\pi/6)\sin(\pi/6))\hat{\boldsymbol{\imath}} - 4\sin(\pi/6)\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}} \\ &= (\sqrt{3}/2 - \pi/12)\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}} \end{aligned}$$

5. Find the directional derivative of $f(x, y) = \frac{x^2}{y^3}$ at $(2, -1)$ in the direction of $1\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}}$. [7 pts]

Solution:

The direction corresponds to the unit vector $\bar{\boldsymbol{u}} = (1\sqrt{2})\hat{\boldsymbol{\imath}} - (1\sqrt{2})\hat{\boldsymbol{\jmath}}$ and so we have:

$$\begin{aligned} D_{\bar{\boldsymbol{u}}} f &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{2x}{y^3}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{3x^2}{y^4}\right) \\ D_{\bar{\boldsymbol{u}}} f(2, -1) &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{4}{-1}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{12}{1}\right) \end{aligned}$$

6. Let $f(x, y, z) = x^3 + 3xy + xe^z$, and suppose $x(t)$, $y(t)$, and $z(t)$ are functions satisfying $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $x'(0) = 1$, $y'(0) = 2$, and $z'(0) = 3$. Calculate the following: [10 pts]

$$\frac{df}{dt} \text{ at } t = 0$$

Solution:

By the chain rule we have:

$$\frac{df}{dx} = (3x^2 + 3y + e^z)x'(t) + (3x)y'(t) + (xe^z)z'(t)$$

Then when $t = 0$ we have $x = 1$, $y = 0$, and $z = 0$ and so:

$$\begin{aligned}\frac{df}{dx}\Big|_{t=0} &= (3(1)^2 + 3(0) + e^0)x'(0) + (3(1))y'(0) + (1e^0)z'(0)\\ &= 4(1) + 3(2) + 1(3)\\ &= 13\end{aligned}$$

7. Suppose $f(x, y) = xy + y^2$. If $\bar{u}$ is a unit vector which makes an angle of $\pi/6$ with ∇f at $(2, -1)$, find $D_{\bar{\boldsymbol{u}}} f(2, -1)$. [10 pts]

   Solution:

   We know that:

$$D_{\bar{\boldsymbol{u}}} f = \bar{\boldsymbol{u}} \cdot \nabla f = ||\bar{\boldsymbol{u}}|| \, ||\nabla f|| \cos\theta = ||\nabla f|| \cos\theta$$

   So then with $\nabla f = y\hat{\boldsymbol{\imath}} + (x + 2y)\hat{\boldsymbol{\jmath}}$:

$$D_{\bar{\boldsymbol{u}}} f(2, -1) = || - 1\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}|| \cos(\pi/6) = \sqrt{3}/2$$

8. All together on one xy-plane sketch the level curves for $f(x, y) = x^2 + y^2$ for $c = 0, 2, 4$. Label each with its value of c. [10 pts]

   Solution:

9. Find all three of the critical points for the function $h(x, y) = x^2y - 2x^2 - y^2$. For each critical point calculate if it is a relative maximum, relative minimum or saddle point. [15 pts]

Solution:

To find the critical points we solve:

$$h_x = 2xy - 4x = 0$$
$$h_y = x^2 - 2y = 0$$

The first is $2x(y - 2) = 0$ so $x = 0$ or $y = 2$.

If $x = 0$ then the second yields $y = 0$ so $(0, 0)$.

If $y = 2$ then the second yields $x^2 = 4$ so $(2, 2)$ and $(-2, 2)$.

We then have:

$$D(x, y) = (2y - 4)(-2) - (2x)^2$$

Testing:

$(0, 0)$: $D(0, 0) = 8$ and $h_{xx}(0, 0) = -2$ so it's a relative maximum.

$(2, 2)$: $D(2, 2) = -16$ so it's a saddle point.

$(-2, 2)$: $D(-2, 2) = -16$ so it's a saddle point.

10. Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y) = xy$ on the circle $x^2 + (y-1)^2 = 1$. [15 pts]

Solution:

We have $g(x, y) = x^2 + (y-1)^2$ and hence we have the system:

$$\begin{aligned} y &= \lambda(2x) \\ x &= \lambda(2(y-1)) \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

We solve for λ in the first two and set them equal:

$$\begin{aligned} \frac{y}{2x} &= \frac{x}{2(y-1)} \\ \frac{y}{x} &= \frac{x}{y-1} \\ x^2 &= y(y-1) \end{aligned}$$

We plug this into the third:

$$\begin{aligned} y(y-1) + (y-1)^2 &= 1 \\ y^2 - y + y^2 - 2y + 1 &= 1 \\ 2y^2 - 3y &= 0 \\ y(2y-3) &= 0 \end{aligned}$$

Thus $y = 0$ or $y = 3/2$.

If $y = 0$ then $x^2 + (0-1)^2 = 1$ and so $x = 0$ yielding $(0, 0)$.

If $y = 3/2$ then $x^2 + (3/2 - 1)^2 = 1$ and so $x^2 = 3/4$ yielding $(\pm\sqrt{3}/2, 3/2)$.

If we test these:

- $f(0, 0) = (0)(0) = 0$
- $f(+\sqrt{3}/2, 3/2) = 3\sqrt{3}/4$ - The maximum!
- $f(-\sqrt{3}/2, 3/2) = -3\sqrt{3}/4$ - The minimum!