MATH 241 Sections 03** Exam 3

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups. Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. (a) Write down a parameterization of the part of the cylinder $x^2 + y^2 = 4$ inside the sphere $x^2 + y^2 + z^2 = 9$ and above the xy-plane. No sketch is required. [3 pts]

Solution:

The cylinder extends from $z = 0$ to the height where the two meet. This occurs when $4 + z^2 = 9$ or $z = \sqrt{5}$. A reasonable parametrization is then:

$$\bar{r}(z, \theta) = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + z\,\hat{k}$$
$$0 \le \theta \le 2\pi$$
$$0 \le z \le \sqrt{5}$$

(b) Write down a parameterization of the part of the plane $2x + y + 3z = 12$ inside the cylinder $x^2 + z^2 = 4$. No sketch is required. [3 pts]

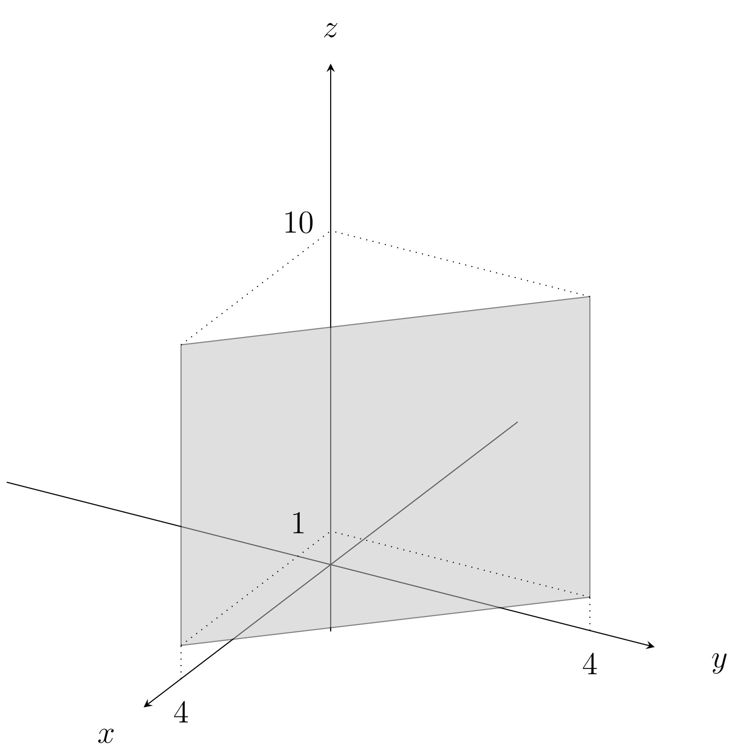
Solution:

$$\bar{r}(r, \theta) = r\cos\theta\,\hat{\imath} + (12 - 2r\cos\theta - 3r\sin\theta) + r\sin\theta\,\hat{k}$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

(c) Sketch the surface parameterized by: [4 pts]

$$\bar{r}(y, z) = (4 - y)\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$
$$0 \le y \le 4$$
$$1 \le z \le 10$$

Solution:



2. **Instruction:**

Let A be the sum of the digits of your UID.
Let B be the largest single digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let R be the filled-in triangle with corners $(0,0)$, $(0,6A)$ and $(1,B)$. [15 pts]

Set up and evaluate the double integral:

$$\iint_R x + 1\, dA$$

You Should Evaluate Your Resulting Integral!

Solution: As a vertically simple region the upper function is $y = (B-6A)x + 6A$ and the lower function is $y = Bx$.

The integral is then:

$$\begin{aligned}
\iint_R x + 1\, dA &= \int_0^1 \int_{Bx}^{(B-6A)x+6A} x + 1\, dy\, dx \\
&= \int_0^1 xy + y\Big|_{Bx}^{(B-6A)x+6A} dx \\
&= \int_0^1 [x((B-6A)x + 6A) + (B-6A)x + 6A] - [x(Bx) + Bx]\ dx \\
&= \int_0^1 Bx^2 - 6Ax^2 + 6Ax + Bx - 6Ax + 6A - Bx^2 - Bx\, dx \\
&= \int_0^1 -6Ax^2 + 6A\, dx \\
&= -2Ax^3 + 6Ax\Big|_0^1 \\
&= 4A
\end{aligned}$$

3. Let R be the region inside $r = \frac{\sqrt{3}}{2}$, outside $r = \cos\theta$, and above the x-axis. Set up but do not evaluate the iterated double integrals (plural!) in polar coordinates for the volume above R and under the paraboloid $f(x, y) = x^2 + y^2$. [10 pts]

You Should Not Evaluate Your Resulting Integrals!

Solution: The functions meet when $\cos\theta = \frac{\sqrt{3}}{2}$ which occurs at $\theta = \frac{\pi}{6}$.

On the interval $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ the near function is $r = \cos\theta$ and the far function is $r = \frac{\sqrt{3}}{2}$.

On the interval $\frac{\pi}{2} \leq \theta \leq \pi$ there is no near function and the far function is $r = \frac{\sqrt{3}}{2}$.

Thus this requires two integrals:

$$\text{Volume} = \iint_R x^2 + y^2 \, dA = \int_{\pi/6}^{\pi/2} \int_{\cos\theta}^{\sqrt{3}/2} r^2 \, r \, dr \, d\theta + \int_{\pi/2}^{\pi} \int_0^{\sqrt{3}/2} r^3 \, dr \, d\theta$$

4. **Instruction:**

Let C be the sum of the first four (leftmost) digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let D be the solid object bounded by the four surfaces: [15 pts]

$$y = C^2 - x^2$$
$$y = x + C$$
$$z = y$$
$$z = 2y$$

If the density at a point is given by the function $f(x, y, z) = z$, set up but do not evaluate the iterated triple integral in rectangular coordinates for the mass of D.

You Should Not Evaluate Your Resulting Integral!

Solution: The sides of the region R are bounded by $y = C^2 - x^2$ and $y = x + C$ which meet at:

$$\begin{aligned} C^2 - x^2 &= x + C \\ x^2 + x + (C - C^2) &= 0 \\ x^2 + x - C(C-1) &= 0 \\ (x + C)(x - (C-1)) &= 0 \end{aligned}$$

Hence at $x = -C$ and $x = C - 1$.

The mass is then:

$$\text{Mass} = \iiint_D z\, dV = \int_{-C}^{C-1} \int_{x+C}^{C^2-x^2} \int_{y}^{2y} z\, dz\, dy\, dx$$

5. Let D be the solid object in the first octant and between the paraboloid $z = x^2 + y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$. [10 pts]

If the density at a point is given by the function $f(x, y, z) = x^2$, set up but do not evaluate the iterated triple integral in cylindrical coordinates for the mass of D.

You Should Not Evaluate Your Resulting Integral!

Solution: In cylindrical the paraboloid is $z = r^2$ and the cone is $z = 2r$. Observing that they meet when:

$$\begin{aligned} r^2 &= 2r \\ r^2 - 2r &= 0 \\ r(r-2) &= 0 \end{aligned}$$

We have $r = 0$ (so $z = 0$) and $r = 2$ (so $z = 4$), thus they meet at the origin and then the cone rises more quickly until they meet again at $z = 4$.

The corresponding R is then the quarter-disk of radius 2 and so we have:

$$\text{Mass} = \iiint_D x^2\, dV = \int_0^{\pi/2} \int_0^2 \int_{r^2}^{2r} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta$$

6. **Instruction:** [15 pts]

Let A be the sum of the digits of your UID.
Let B be the largest single digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

The following sum of two double integrals is impossible as given. Re-iterate as necessary to just one double integral which can be evaluated.

$$\int_0^A \int_{\frac{B}{2A}x}^{\frac{B}{A}x} \cos(y^2)\,dy\,dx + \int_A^{2A} \int_{\frac{B}{2A}x}^{B} \cos(y^2)\,dy\,dx$$

You Should Evaluate Your Resulting Integral!

Hint: Draw a picture!

Solution: The integrals make up two vertically simple regions which together form one horizontally simple region. Re-iterating as horizontally simple results in:

$$\begin{aligned}
\int_0^A \int_{\frac{B}{2A}x}^{\frac{B}{A}x} \cos(y^2)\,dy\,dx + \int_A^{2A} \int_{\frac{B}{2A}x}^{B} \cos(y^2)\,dy\,dx &= \int_0^B \int_{\frac{A}{B}y}^{\frac{2A}{B}y} \cos(y^2)\,dx\,dy \\
&= \int_0^B x\cos(y^2)\Big|_{\frac{A}{B}y}^{\frac{2A}{B}y}\,dy \\
&= \int_0^B \frac{2A}{B}y\cos(y^2) - \frac{A}{B}y\cos(y^2)\,dy \\
&= \int_0^B \frac{A}{B}y\cos(y^2)\,dy \\
&= \frac{A}{2B}\sin(y^2)\Big|_0^B \\
&= \frac{A}{2B}\sin(B^2)
\end{aligned}$$

7. Let D be the solid object inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ [10 pts]

Write down an iterated triple integral in spherical coordinates which yields the volume of D.

You Should Not Evaluate Your Resulting Integral!

Solution:

The sphere has equation $\rho = 4$ and the cylinder has equation $\rho^2 \sin^2 \phi = 4$, or $\rho = 2 \csc \phi$.

These meet when $2 \csc \phi = 4$ or $\sin \phi = 2$ which occurs at $\phi = \frac{\pi}{6}$ and $\phi = \frac{5\pi}{6}$.

Thus the integral is:

$$\iiint_D 1 \, dV = \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{2 \csc \phi}^{4} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

8. **Instruction:** [15 pts]

Let S be the smallest nonzero digit of your UID.
Let L be the largest digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Suppose R is the four-sided figure with corners $(0, L)$, $(0, 2L)$, $(S, 0)$ and $(2S, 0)$.

Apply the change of variables $u = Lx + Sy$ and $v = Lx - Sy$ to the following integral:

$$\iint_R \frac{Lx - Sy}{Lx + Sy}$$

and rewrite the resulting integral as an iterated double integral in the uv-plane.

You Should Not Evaluate Your Resulting Integral!

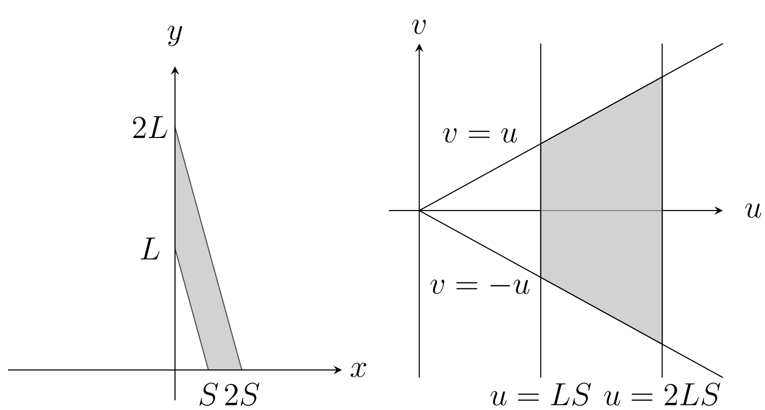
Solution:

The edges of R are the diagonal lines $y = -\frac{L}{S}x + 2L$ and $y = -\frac{L}{S} + L$ and the axes $x = 0$ and $y = 0$.

The diagonal lines can be rewritten as $Lx + Sy = 2LS$ and $Lx + Sy = LS$ respectively so with the substitution they become $u = 2LS$ and $u = LS$.

The changes of variables add to yield $u + v = 2Lx$ so that $x = \frac{u+v}{2L}$ and subtract to yield $u - v = 2Sy$ so that $y = \frac{u-v}{2S}$. Thus the axes edges become $\frac{u+v}{2L} = 0$, or $u + v = 0$, and $\frac{u-v}{2S} = 0$, or $u - v = 0$.

The two regions are shown here:



The Jacobian is then:

$$\begin{vmatrix} \frac{1}{2L} & \frac{1}{2L} \\ \frac{1}{2S} & -\frac{1}{2S} \end{vmatrix} = -\frac{1}{2LS}$$

So the final integral is:

$$\iint_S \frac{v}{u} \left| -\frac{1}{2LS} \right| \, dA = \int_{LS}^{2LS} \int_{-u}^{u} \frac{v}{u} \left(\frac{1}{2LS} \right) \, dv \, du$$